Banks' Liquidity Management and Financial Fragility*

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Abstract

How do banks manage liquidity against financial fragility? To answer this question, we study an economy where banks control both sides of their balance sheets, and undertake maturity transformation to insure depositors against idiosyncratic and aggregate uncertainty. Moreover, strategic complementarities might trigger depositors' self-fulfilling runs, modeled as "global games". If depositors' risk aversion is sufficiently high, the banks engage either in liquidity hoarding if the productive asset in portfolio is sufficiently liquid, or in liquidity cushioning if it is sufficiently illiquid. Ex ante, if the probability of the idiosyncratic shock is sufficiently large, banks hold extra precautionary liquidity, and narrow banking is not viable.

Keywords: banks, liquidity, financial fragility, self-fulfilling runs, global games

JEL codes: G01, G21, G28

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1 Introduction

It is a well documented fact that banks hold large amounts of liquidity at times of financial distress. As an example, Figure 1 shows the evolution of the sum of (i) the total excess reserves held by the banks subject to minimum reserve requirements in the Euro Area, and (ii) the size of the Eurosystem's deposit facility: It peaked at around EUR250 Billion during the 2007-2009 global financial crisis, and at around EUR300 Billion and EUR800 Billion during the 2010-2012 EU joint bank and sovereign crisis. Many explanations have been proposed for the observed link between bank liquidity and financial distress, including precautionary savings (Ashcraft et al., 2011; Acharya and Merrouche, 2013) and counterparty risk (Caballero and Simsek, 2013; Heider et al., 2015), all based on the assumption that banks face fundamental uncertainty against which they might want to hold safe assets. However, there is also an extensive evidence showing that banks are prone to financial fragility induced by the depositors' self-fulfilling expectations of crises. Indeed, the very essence of banking, i.e. liquidity and maturity transformation, creates financial fragility through a mismatch in banks' balance sheets that leads to depositors' self-fulfilling runs. Financial fragility and self-fulfilling runs are not a phenomenon of the past: For example, Argentina in 2001 and Greece in 2015 faced such systemic events. On top of that, there is a wide consensus that both the 2007-2009 global financial crisis and the 2010-2012 EU joint bank and sovereign crisis had a significant self-fulfilling component (Gorton and Metrick, 2012; Baldwin et al., 2015). These considerations call for a theory of the interaction between banks' liquidity management and self-fulfilling financial fragility. This is the aim of the present paper.

The distinctive feature of our argument is that the interaction between liquidity management and financial fragility goes in both directions. In fact, on the one hand, financial fragility has a non-trivial effect on banks' liquidity management ex ante, as they anticipate that they can pay excessive withdrawals either by rolling over liquidity or by liquidating the more productive assets on their balance sheets. On the other hand, liquidity management affects investors' perception of how resilient banks are to fundamental uncertainty, which feeds back into financial fragility. Accordingly, we propose a theory of banking, in which banks are exposed to idiosyncratic uncertainty in the form of liquidity shocks that force their depositors to withdraw in an interim period (i.e. before the maturity of a productive investment), and aggregate uncertainty in the form of productivity shocks.

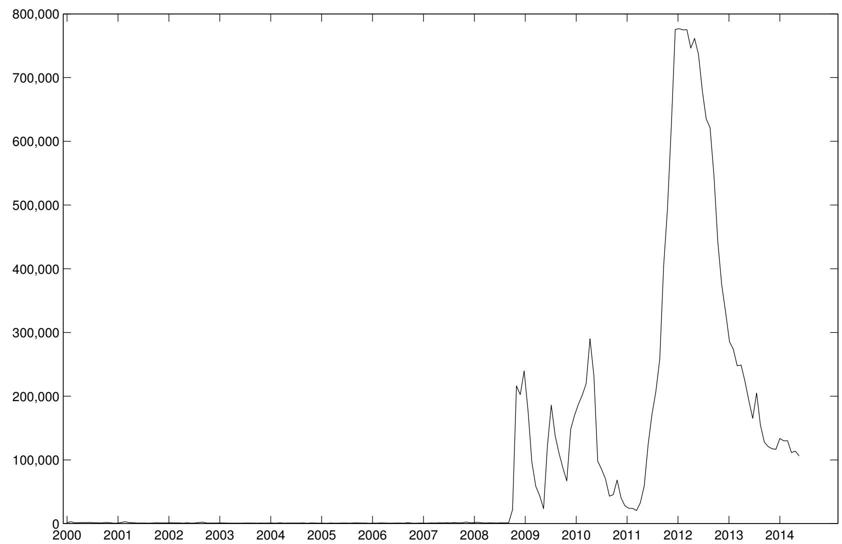

Figure 1: The sum of (i) the total excess reserves held by the banks subject to minimum reserve requirements in the Euro Area, and (ii) the size of the Eurosystem's deposit facility (end of month, millions of euros). Source: European Central Bank.

Banks also face financial fragility, due to incomplete contractibility related to the idiosyncratic liquidity shocks and imperfect information about the aggregate productivity shocks. This leads to multiple equilibria, with the possibility of self-fulfilling runs by the banks' depositors in the interim period, due to strategic complementarities in their withdrawal decisions.[1] We resolve the multiplicity of equilibria following the "global game" approach by Carlsson and van Damme (1993) and Morris and Shin (1998).

To hedge against both fundamental (i.e. idiosyncratic and aggregate) and self-fulfilling uncertainty while undertaking productive maturity transformation, the banks collect deposits, and invest in liquidity and in a partially illiquid but productive asset. Characterizing both sides of banks' balance sheets complicates the analysis in a substantial way, which represents the methodological contribution of the paper. More importantly, it enables us to offer a complete analysis of banks' liquidity management from an ex-ante perspective, i.e. in anticipation of fundamental and

[1]For this argument to hold, we need to assume that there exists no deposit insurance and that no government can credibly commit to suspend convertibility in the case of a run. These assumptions find their justification in the growing role played by uninsured bank deposits and the shadow banking system, that offers bank services – and in particular liquidity and maturity transformation – without any regulation or government assistance (Pozsar et al., 2010).

self-fulfilling uncertainty, as well as from an ex-post perspective, when the latter may materialize. Moreover, in this framework both the concept of precautionary liquidity and extra precautionary liquidity are well-defined: the former, by comparing an economy with both idiosyncratic and aggregate uncertainty (but without self-fulfilling uncertainty, due to the presence of perfect information) to one with idiosyncratic uncertainty alone; the latter, by adding self-fulfilling uncertainty.

On top of the choice of the deposit contract and asset portfolio, we further allow the banks to choose a pecking order, between deploying liquidity and liquidating the productive asset, that they follow in the interim period to satisfy depositors' withdrawals. Including the pecking order as part of the strategies of the banks is crucial because it influences the depositors' withdrawal decisions in the interim period. That is because, on the one hand, deploying liquidity comes at the cost of reducing insurance against aggregate uncertainty. On the other hand, liquidating the productive asset is also costly, in terms of forgone resources due to illiquidity and forgone future consumption.

We characterize the unique symmetric equilibrium of the game between banks and depositors. If only the depositors hit by the idiosyncratic liquidity shocks withdraw in the interim period, no run materializes: The banks are solvent, and use only liquidity to serve their depositors' withdrawals without liquidating the productive asset. If instead all depositors withdraw in the interim period, a run materializes: The banks are insolvent and are forced to liquidate all the productive assets in portfolio and close down. These are the only two final outcomes observable in the economy, and in both cases the pecking order is immaterial: in the first one, because the banks choose to not liquidate, and in the second because they are forced to do it. Still, there exists the possibility of partial depositors' runs, in which banks are illiquid but solvent. This case is "off equilibrium" because we focus our attention on a depositors' symmetric choice: In other words, either all depositors run, or no one runs. That is the reason why the choice of the pecking order in this off-equilibrium situation is apparently inconsequential. However, it affects how the depositors form their expectations of a run, and as a consequence self-fulfilling uncertainty. This in turn has an effect on both sides of banks' balance sheets, and therefore on the allocation "on equilibrium". Our first result shows that the off-equilibrium pecking order depends on the depositors' relative risk aversion and on the illiquidity of the productive asset. If the depositors' relative risk aversion is sufficiently high, the banks first liquidate the productive asset and then deplete liquidity (i.e. they engage in liquidity "hoarding") if the former is sufficiently liquid. Differently, they first deplete liquidity and then liqui-

Idiosyncratic shocks	+	Aggregate shocks	+	Self-fulfilling runs
$\Downarrow$		$\Downarrow$		$\Downarrow$
Liquidity	$<$	Precautionary liquidity	$<$	Extra precautionary liquidity

Figure 2: Sources of uncertainty and liquidity in banks' asset portfolios.

date the productive asset (i.e. they engage in liquidity "cushioning") when the latter is sufficiently illiquid.

Our second result characterizes banks' liquidity management ex ante, i.e. in anticipation of fundamental and self-fulfilling uncertainty, and the conditions under which banks build up precautionary liquidity and eventually extra precautionary liquidity in their asset portfolios. As summarized in Figure 2, precautionary liquidity is the way through which banks save against aggregate uncertainty over and above the liquidity that they need to insure their depositors against idiosyncratic uncertainty. On top of precautionary liquidity, the anticipation of a self-fulfilling run imposes a distortion in banks' asset portfolios, that might force them to further increase liquidity and lower insurance against idiosyncratic uncertainty. We show that this is indeed the case if the depositors are sufficiently likely to suffer idiosyncratic uncertainty. In other words, banks hold extra precautionary liquidity in the presence of financial fragility, in the sense that they further increase liquidity above what they would need against fundamental uncertainty alone.

Finally, we study liquidity regulation. In particular, we analyze banks' behavior under a requirement that forces them to be "narrow", i.e. such that they hold sufficient liquidity to pay all their depositors' withdrawals in the interim period, even in the case of a run. Interestingly, as the proposed environment allows a full characterization of both sides of their balance sheets, we are able to study how the fulfillment of liquidity regulation distorts banks' liquidity management on the asset side as well as the deposit contract on the liability side. In particular, we find that a narrow bank in equilibrium holds just enough liquidity to become run proof. Moreover, under some mild conditions it chooses to be fully liquid and to offer no insurance against idiosyncratic uncertainty. In turn, this might undermine the viability of banking itself. In fact, narrow banking is not viable if the depositors are sufficiently likely to face idiosyncratic uncertainty: In that case, a bank would provide the same allocation that the depositors could reach under autarky (i.e. without banks), and that would make it at most redundant.[2]

Contribution to the literature The present paper contributes to the literature on banks' liquidity and financial fragility by developing a novel framework in which banks hold liquidity against both aggregate uncertainty and self-fulfilling runs. In fact, in the first-generation models of bank runs, Cooper and Ross (1998) and Ennis and Keister (2006) study banks' liquidity management in an environment à la Diamond and Dybvig (1983) with self-fulfilling runs, but without aggregate uncertainty. In there, the depositors run because of the realization of an exogenous "sunspot", and banks hold extra liquidity in equilibrium, but only to be able to serve all depositors in the case of a run, i.e. to be run-proof. In other words, contrary to the empirical evidence, in equilibrium these models do not exhibit extra liquidity and self-fulfilling runs simultaneously. Allen and Gale (1998) instead study banks' liquidity management in an environment with fundamental uncertainty but no self-fulfilling runs. In their model, the banks do not hold extra liquidity in equilibrium, but offer a standard deposit contract coupled with default in the bad states of the world, thus allowing optimal risk sharing. Similarly, Gale and Yorulmazer (2013) study banks' liquidity management both for precautionary reasons (i.e. to hedge against fundamental shocks) and speculative reasons (i.e. to take advantage of fire sales) but again do not analyze self-fulfilling runs.

In the second-generation models of bank runs (Rochet and Vives, 2004; Goldstein and Pauzner, 2005) the economy does feature aggregate uncertainty, and the equilibrium selection mechanism in the presence of multiple equilibria is endogenous via the introduction of a global game. Yet, in these models there is often no liquidity, and as a consequence no role for liquidity regulation. In Goldstein and Pauzner (2005) this happens because the investment in liquidity is dominated by the one in productive assets, which the banks can liquidate at zero cost. Differently from them, by assuming costly liquidation we are able to meaningfully introduce liquidity, and study maturity transformation and the distortions arising from inefficient liquidation. In Rochet and Vives (2004) and Vives (2014) instead banks do hold liquidity and productive assets in their portfolios, but the structure of their balance sheets and the pecking order during a self-fulfilling run are exogenous. Moreover, even when the balance-sheet structure is endogenized with the introduction of a moral-

[2] This result is reminiscent of Wallace (1996). However, Wallace's argument is based on showing that the feasible allocations under narrow banking are also feasible under autarky. Here instead we directly prove the equivalence of the two equilibrium allocations.

hazard problem on the part of the banks, the constrained-efficient level of banks' liquidity is zero, as a lender of last resort could provide liquidity to the banks at zero costs. Kashyap et al. (2017) develop a model of both sides of banks' balance sheets like we do. However, differently from our work, in their framework technology and timing do not allow the banks to hold extra precautionary liquidity against self-fulfilling uncertainty. Ahnert and Elamin (2014) also study banks' portfolio choice, but some ad-hoc assumptions on technology, timing and information make their work complementary to ours. In particular, differently from us they assume that the banks can liquidate the productive asset at zero costs in the interim period, but do not have any liquidity available in the initial period, thus neglecting the issue of the composition of banks' asset portfolios and the initial liquidity management. Moreover, the banks have access to liquidity only after the depositors' decisions to withdraw and eventually run, and after receiving a perfectly informative signal about the realization of the aggregate state of the economy. As a consequence, in their framework banks merely insure depositors against fundamental uncertainty (and only after a run), do not choose a pecking order, and do not hold extra precautionary liquidity against self-fulfilling uncertainty in the initial period.

More generally, the present paper contributes to the analysis of liquidity management in financial institutions subject to self-fulfilling uncertainty.[3] Chen et al. (2010) empirically analyze the presence of strategic complementarities among mutual funds' investors. Their main finding is that the sensitivity of a fund's outflows to bad performance is stronger in funds that invest in more illiquid assets than in funds that invest in less illiquid assets. To rationalize this, the authors develop a global-game theory of strategic complementarities, and show that the threshold signal for a self-fulfilling run is decreasing in the liquidity of a fund's assets. This holds true also in our environment, yet we further prove that the threshold signal is a convex function of a fund's asset liquidity, and this is a key property to endogenize the pecking order. The additional empirical evidence on mutual funds' liquidity management is instead not conclusive. On the one hand, Chernenko and Sunderam (2016) find that mutual funds engage in liquidity "cushioning", i.e. a partial use of cash holdings to manage unexpected outflows. On the other hand, Morris et al. (2017) find evidence of liquidity "hoarding", i.e. mutual funds managing unexpected outflows by selling the underlying assets in their portfolios while retaining liquidity. Our characterization of the endogenous pecking

[3] A few other examples of this literature also include Goldstein et al. (2017) on corporate bond funds, and Schmidt et al. (2016) on money market mutual funds.

order offers a reconciliation of this contrasting results, based on asset illiquidity and risk aversion.[4] Finally, Liu and Mello (2011) study the liquidity management of hedge funds facing coordination risk. To this end, they develop a global game in which the fund's investors decide whether to redeem early their investment. Differently from them, the focus of the present paper is on banks, and their maturity transformation and fragility in the presence of risk-averse depositors. Accordingly, we analyze banks' liquidity management against both aggregate uncertainty and self-fulfilling runs. Moreover, we characterize the equilibrium deposit contract and the optimal pecking order, which Liu and Mello (2011) instead leave as exogenous.

Outline The rest of the paper is organized as follows. In section 2, we lay down the basic features of the environment. In section 3, we study the withdrawing decisions of the depositors. In section 4, we characterize the banking equilibrium. In section 5, we study the effect of liquidity regulation. Finally, section 6 concludes.

2 Environment

The economy lives for three periods, labeled $t = 0, 1, 2$, and is populated by a unitary continuum of ex-ante identical agents, all endowed with 1 unit of a consumption good at date 0, and 0 afterwards. At date 1, all agents are hit by a privately-observed idiosyncratic liquidity shock θ, taking value 0 with probability λ and 1 with probability $1 - \lambda$. The law of large numbers holds, hence the probability distribution of the idiosyncratic liquidity shocks is equivalent to their cross-sectional distribution: At date 1, there is a fraction λ of agents in the whole economy whose realized shock is $\theta = 0$, and a fraction $1 - \lambda$ whose realized shock is $\theta = 1$. The idiosyncratic liquidity shocks affect the point in time when the agents want to consume, according to the welfare function $U(c_1, c_2, \theta) = (1 - \theta)u(c_1) + \theta u(c_2)$. In other words, those agents receiving a shock $\theta = 0$ are only willing to consume at date 1, and those receiving a shock $\theta = 1$ are only willing to consume at date 2. Thus, in line with the literature, we refer to them as early consumers and late consumers, respectively. The utility function $u(c)$ is increasing, strictly concave and twice-continuously differentiable, and is such that $u(0) = 0$ and the coefficient of relative risk aversion is strictly larger than 1. Unless otherwise

[4] Jiang et al. (2017) show that corporate-bond mutual funds, in order to meet their investors' withdrawals, move from liquidity cushioning to liquidity hoarding in the time series depending on aggregate market uncertainty. In contrast, our story revolves around asset illiquidity and depositors' risk aversion in the cross section, and focuses only on self-fulfilling withdrawals.

stated, the utility function is the CRRA $u(c) = ((c+\psi)^{1-\gamma} - \psi^{1-\gamma})/(1-\gamma)$. The constant ψ is arbitrarily close to but larger than 0, and can be interpreted as a consumption that the depositors enjoy outside the banking system. The fact that ψ is arbitrarily close to but larger than 0 ensures that $u(0) = 0$, and that the coefficient of relative risk aversion is constant and equal to γ. This also implies that $\lim_{c\to 0} u'(c) = \psi^{-\gamma}$, which is arbitrarily large but finite. In other words, this functional form satisfies the Inada conditions: $\lim_{\psi\to 0^+} \lim_{c\to 0} u'(c) = +\infty$ and $\lim_{c\to +\infty} u'(c) = 0$.

There are two technologies available in the economy to hedge against the idiosyncratic liquidity shocks. The first one is a storage technology, here called "liquidity", yielding 1 unit of consumption at date $t+1$ for each unit invested in t. The second one is instead a productive asset that, for each unit invested at date 0, yields a stochastic return Z at date 2. This stochastic return takes values $R > 1$ with probability p, and 0 with probability $1-p$. The probability of success of the productive asset p represents the aggregate state of the economy, and is uniformly distributed over the interval $[0,1]$, with $\mathbb{E}[p]R > 1$. Moreover, the productive asset can be liquidated at date 1 via a liquidation technology, that allows to recover $r < 1$ units of consumption for each unit liquidated. Intuitively, this means that the economy features a liquid asset, with low but safe yields, and a partially illiquid asset, that yields a low return in the short run, but a possible high return in the long run, subject to the realization of an aggregate productivity shock.

The economy is also populated by a large number of banks, operating in a perfectly-competitive market with free entry. The banks collect the endowments of the agents in the form of deposits, and invest them so as to maximize their profits, subject to agents' participation. Perfect competition and free entry ensure that the banks solve the equivalent dual problem of maximizing the expected welfare of the agents/depositors, subject to their budget constraint. To this end, they offer a deposit contract $\{c, c_L(Z)\}$, stating the amount of early consumption c that the depositors can withdraw at date 1, and the amount $c_L(Z)$ that they can withdraw at date 2.[5] The early consumption c must be lower than what a late consumer would receive if only λ depositors withdraw at date 1, otherwise they would all withdraw at date 1. To repay the depositors according to the deposit contract, the banks at date 0 invest the deposits – which are the only liability on their balance sheets – in a

[5]In order to rule out uninteresting run equilibria, the amount of early consumption c must be smaller than $\min\{1/\lambda, R\}$. The fact that the banks have to offer a standard deposit contract here is assumed. In a Diamond-Dybvig environment, Farhi et al. (2009) show that a standard deposit contract with an uncontingent amount of early consumption endogenously emerge in equilibrium in the presence of non-exclusive contracts.

portfolio of L units of liquidity and $1 - L$ units of the productive asset. Then, given the deposit contract and asset portfolio, the banks at date 1 pay c to all the depositors who withdraw early, until their resources are exhausted.[6] Finally, at date 2 the depositors who have not withdrawn at date 1 are residual claimants of an equal share of the remaining resources. When resources are exhausted at date 1, and the banks are not able to fulfill their contractual obligations with the depositors anymore, they go into bankruptcy. In this case, they are forced to liquidate all the productive assets in portfolio, and serve the depositors according to an "equal service constraint", i.e. such that all of them get an equal share of the available resources.

At date 0, after they choose the deposit contract and asset portfolio, the banks also choose the "pecking order" with which to use the available assets in order to pay early withdrawals at date 1. This timing accounts for the fact that a bank takes more time to change its portfolio strategy and liquidate the productive asset than a depositor to implement his or her withdrawal strategy.[7] For the same reason, we also assume that banks can choose only one pecking order.[8] Under the pecking order {Liquidation, Liquidity} the banks first liquidate the productive asset and then deploy liquidity. Under the pecking order {Liquidity, Liquidation} they instead first deploy liquidity and then liquidate the productive asset. As at bankruptcy a bank is forced to liquidate all the productive assets in portfolio, the choice of the pecking order in this case is immaterial.

We assume that the depositors cannot observe the true value of the realization of the aggregate state of the economy p, but receive at date 1 a signal $\sigma = p + e$ about it. The term e is an idiosyncratic noise, that is indistinguishable from the true value of p and uniformly distributed over the interval $[-\epsilon, +\epsilon]$, where ϵ is positive but arbitrarily close to zero. Given the received signal, a late consumer decides whether to wait and withdraw from the bank at date 2, as the realization of the idiosyncratic shock would command, or "run on the bank" and withdraw at date 1. This decision is based on the expected advantage of waiting versus running, according to the scheme to be described in section 3 that explicitly depends on the offered deposit contract. On top of that,

[6] We abstract from the possibility that a government suspends deposit convertibility. Such a government intervention is time inconsistent, and therefore does not resolve the fragility of banks (Ennis and Keister, 2009).

[7] This assumption about the timing of actions is not restrictive: Assuming that the banks take the decision about the pecking order at the same time as the deposit contract and asset portfolio would not alter the characterization of the equilibrium.

[8] The analysis of the case in which the banks choose a pecking order for any possible realization of the number of early withdrawers would seriously complicate the tractability of the problem, and goes beyond the scope of the paper.

t=0	t=1	t=2
(i) Banks collect the deposits, and choose the deposit contract $\{c, c_L(Z)\}$ and asset portfolio $\{L, 1\text{-}L\}$; (ii) Banks choose the pecking order.	(i) Private types and signals are revealed; (ii) Early consumers withdraw, and late consumers decide whether to run.	(i) Late consumers who have not run withdraw an equal share of the available resources left.

Figure 3: The timing of actions.

the withdrawal decision also depends on the banks' pecking-order decision at date 0, as that will ultimately affect the available resources at date 1 (due to costly liquidation) as well as date 2 (due to the uncertainty about the return on the productive asset).

Figure 3 shows the timing of actions. At date 0, the depositors deposit their endowments in the banks, and the banks choose the deposit contract $\{c, c_L(Z)\}$, the asset portfolio $\{L, 1-L\}$ and the pecking order that they adopt to manage liquidity at date 1. At date 1, all depositors get to know their private types and signals, and the early consumers withdraw, while the late consumers, once observed their own signals, decide whether to run or not. Finally, at date 2 those late consumers who have not withdrawn at date 1 withdraw an equal share of the available resources left. We solve the model by backward induction, and characterize a pure-strategy symmetric perfect Bayesian equilibrium.[9] Hence, we focus our attention on the behavior of a representative bank. The definition of equilibrium is the following:

Definition 1. *Given the distributions of the idiosyncratic liquidity shocks θ, of the aggregate productivity shock Z and of the individual signals σ, a banking equilibrium is a deposit contract $\{c, c_L(Z)\}$, an asset portfolio $\{L, 1-L\}$, a pecking order and depositors' withdrawal decisions such that:*

- *The depositors' withdrawal decisions maximize their expected welfare;*
- *The pecking order, the deposit contract and the asset portfolio maximize the depositors' expected welfare, subject to the bank's budget constraints;*

[9] Introducing asymmetric equilibria in a global game (like for example in the environment with heterogeneous agents by Drozd and Serrano-Padial (2018)) goes beyond the scope of this paper.

- *The bank's and depositors' beliefs are updated according to the strategies employed and the Bayes rule.*

2.1 Autarkic equilibrium

As a benchmark to study the viability of the banking equilibrium in the incoming sections, we start our analysis with the characterization of the equilibrium in autarky. Assume that the agents cannot access the banking system at date 0, but can invest in a portfolio of liquidity L and productive assets $1-L$, in anticipation of the idiosyncratic liquidity shock θ and of the aggregate productivity shock Z. Then, if an agent turns out to be an early consumer, he or she will consume the liquidation value of his or her asset portfolio, namely $c^A = L + r(1-L)$, which is clearly lower than or equal to 1 as it is a linear combination of 1 and $r < 1$. If instead the agent turns out to be a late consumer, he or she will consume an amount which depends on the realization of the productivity shock Z plus the amount of liquidity which is rolled over to date 2, i.e. $c_2^A(R) = R(1-L) + L$ or $c_2^A(0) = L$. Then, at date 0, the portfolio problem boils down to:

$$\max_L \ \lambda u(L + r(1-L)) + (1-\lambda)\int_0^1 \Big[pu(R(1-L)+L) + (1-p)u(L)\Big]dp, \tag{1}$$

subject to $L \leq 1$. Attach the Lagrange multiplier χ to the last constraint. The first-order condition of the problem reads:[10]

$$\lambda(1-r)u'(L + r(1-L)) = (1-\lambda)\mathbb{E}[p]\Big[u'(R(1-L)+L)(R-1) - u'(L)\Big] + \chi. \tag{2}$$

It can be proved that, if the condition:

$$\frac{\lambda(1-r)}{1-\lambda} < \mathbb{E}[p](R-2) \tag{3}$$

holds, the equilibrium amount of liquidity L^A is smaller than 1. To see that, notice that if $L^A = 1$ the equilibrium condition would yield a Lagrange multiplier:

$$\chi = \Big[\lambda(1-r) - (1-\lambda)\mathbb{E}[p](R-2)\Big]u'(1). \tag{4}$$

[10] In equilibrium L must be positive, as $L = 0$ would not satisfy the first-order condition because of the Inada conditions.

Under condition (3), this expression is negative, which is impossible by the definition of Lagrange multiplier. Hence, we prove the following:

Lemma 1. *If Condition* (3) *holds, the autarkic equilibrium is characterized by:*

$$\lambda(1-r)u'(L^A + r(1-L^A)) + (1-\lambda)\mathbb{E}[p]u'(L^A) = (1-\lambda)\mathbb{E}[p](R-1)u'(R(1-L^A)+L^A). \quad (5)$$

If λ is sufficiently large so that Condition (3) *does not hold, the autarkic equilibrium yields $L^A = c^A = c_L^A(0) = c_L^A(R) = 1$.*

Proof. In the text above. ■

Intuitively, (5) shows that an agent in autarky chooses an equilibrium asset portfolio such that the expected marginal benefits of holding liquidity, in terms of early consumption and late consumption in the bad state of the world (as $c_L^A(0) = L^A$), must be equal to the expected marginal costs of holding liquidity, in terms of late consumption $c_L^A(R)$ lost in the good state of the world. Yet, if the probability of the idiosyncratic shock is so high that it prevails over the investment loss from not investing in the productive asset, the agent chooses in equilibrium a fully liquid asset portfolio. For the remaining part of the paper, we assume that this is the case, and Condition (3) does not hold.

2.2 Equilibrium with perfect information

As a further benchmark, here we characterize a banking equilibrium in which the representative bank is perfectly informed about depositors' types, i.e. it can observe the realization of the idiosyncratic liquidity shocks hitting the depositors (but not the realization of the aggregate state) and maximizes their expected welfare subject to budget constraints. More formally, the bank solves:

$$\max_{c, c_L(Z), L, D} \lambda u(c) + (1-\lambda)\mathbb{E}[u(c_L(Z))], \quad (6)$$

subject to the budget constraints:

$$L + rD \geq \lambda c, \quad (7)$$

$$(1-\lambda)c_L(Z) + \lambda c = Z(1-L-D) + L + rD, \quad (8)$$

where the last constraint has to hold for any $Z \in \{0, R\}$, and to the non-negativity constraint $D \geq 0$.[11] At date 0, the bank collects all endowments, and invests them in an amount L of liquidity and $1 - L$ of productive assets. At date 1, the liquidity constraint (7) states that the amount of liquid assets, given by the sum of liquidity plus the resources generated by liquidating an amount D of productive assets at rate r, must be sufficient to pay early consumption c to the λ early consumers. Any resource $L + rD - \lambda c$ left constitutes precautionary liquidity, and is rolled over to date 2. The precautionary liquidity, together with the return from the remaining productive assets, pay late consumption:

$$c_L(Z) = \frac{Z(1 - L - D) + L + rD - \lambda c}{1 - \lambda} \tag{9}$$

for any realization of the aggregate productivity shock Z.

Plugging the budget constraints in the objective function, the bank's problem reads:

$$\max_{c,L,D} \lambda u(c) + (1 - \lambda) \int_0^1 \left[pu \left(\frac{R(1 - L - D) + L + rD - \lambda c}{1 - \lambda} \right) + (1 - p)u \left(\frac{L + rD - \lambda c}{1 - \lambda} \right) \right] dp, \tag{10}$$

subject to the liquidity constraint $L + rD \geq \lambda c$ and $D \geq 0$. In this framework, we can prove the following:

Lemma 2. *The banking equilibrium with perfect information exhibits no liquidation of the productive asset ($D^{PI} = 0$) and precautionary liquidity ($L^{PI} > \lambda c^{PI}$). The deposit contract and asset portfolio satisfy the Euler equation:*

$$u'(c^{PI}) = \mathbb{E}[p] R u' \left(\frac{R(1 - L^{PI}) + L^{PI} - \lambda c^{PI}}{1 - \lambda} \right). \tag{11}$$

Moreover, if λ is sufficiently large, the equilibrium deposit contract satisfies:

$$0 < c_L^{PI}(0) < 1 < c^{PI} < c_L^{PI}(R). \tag{12}$$

Proof. In Appendix A. ■

The Lemma shows that liquidating the productive asset to create liquidity at date 1 is never

[11] The non-negativity constraints on the other choice variables are always satisfied in equilibrium, given the assumption that the Inada conditions hold.

part of an equilibrium with perfect information, because the recovery rate $r < 1$ implies that the liquidation of the productive asset is too costly. As a consequence, with perfect information the choice of the pecking order is immaterial, which is a result that is going to be relevant for the characterization of the endogenous pecking order in section 4.1. If the probability λ of a depositor being hit by the idiosyncratic liquidity shock is sufficiently large, the bank provides insurance against it by transferring part of the available resources from late consumption to early consumption. Moreover, the bank also provides insurance against the aggregate productivity shock Z by engaging in precautionary savings, i.e. by holding extra liquidity on top of the one needed to cover early consumption and insure against the idiosyncratic liquidity shock. In equilibrium, the bank achieves these objectives by choosing an asset portfolio according to an Euler equation, i.e. so that the marginal rate of substitution between early and late consumption is equal to the expected marginal rate of transformation of the productive asset. Finally, the concavity of the utility function and the assumption that $\mathbb{E}[p]R > 1$ imply that at the equilibrium allocation $c \leq c_L^{PI}(R)$ is satisfied with a strict inequality.

How does the banking equilibrium compare with the autarkic equilibrium? Remember that, if the probability of the idiosyncratic shock is sufficiently large, the agents in autarky choose a fully liquid asset portfolio, and the equilibrium allocation is $c^A = c_L^A(0) = c_L^A(R) = 1$. Then, $c^{PI} > c^A$ means that the bank by pooling risk is able to provide to the depositors better insurance against idiosyncratic uncertainty than what they would get in autarky. In contrast, as $c_L^{PI}(0) < c_L^{PI}(R)$, consumption volatility at date 2 is higher in the banking equilibrium than in autarky. This means that, despite the fact that the agents in autarky completely lose the opportunity to invest in the productive asset, they might still be better off than in the banking equilibrium, especially if they are sufficiently risk averse. However, in the banking equilibrium the bank can always choose to invest all deposits in liquidity, as the agents do in autarky. Put differently, the autarkic allocation is feasible for the bank, but is not chosen. Then, as perfectly competitive banks maximize the expected welfare of the depositors, this must mean that the banking equilibrium with perfect information Pareto-dominates autarky even in the presence of a more volatile consumption profile.

3 Strategic complementarities

We now move to the analysis of the competitive banking equilibrium. As stated above, we characterize it by backward induction, hence in this section we start by studying the withdrawing decisions of a late consumer (as an early consumer withdraws for sure at date 1) who chooses whether to withdraw at date 1 (i.e. "run") or wait until date 2 under the two pecking orders separately. Then, in the following section, we characterize the equilibrium pecking order, and the deposit contract and the choice of the asset portfolio.

Assume that the depositors arrive at the bank at date 1 in random order, and do not know how many of them are in line. As a result, the depositors do not accept a contract contingent on either their position in line or the number of early withdrawals. Due to the commitment to pay an amount of early consumption c, the bank must use liquidity and liquidate the productive asset (in accordance with the chosen pecking order) to pay early withdrawals until the resources are exhausted. As a consequence, if a late consumer expects only the early consumers to withdraw at date 1, he or she will withdraw at date 2 and receive $c_L(R) > c$. However, if a late consumer expects all the other depositors to withdraw at date 1, he or she will rather withdraw at date 1 as well, because in that case he or she will be served pro-rata at date 1 instead of getting zero at date 2. This means that this economy, as any Diamond-Dybvig environments, features a "no run" equilibrium and a "run" equilibrium.

We resolve this multiplicity of equilibria employing the global-game techniques. Each late consumer acts based on his or her private signal σ at date 1, and takes as given the deposit contract and asset portfolio, fixed at date 0, and the pecking order, fixed at date 1 before the signal is realized. Based on this information, he or she creates posterior beliefs about the probability of the realization of the aggregate productivity shock Z and about how many depositors are withdrawing at date 1 (call this number n), and decides whether to withdraw or not. We assume the existence of two regions of extremely high and extremely low signals, where the decision of a late consumer is independent of his or her posterior beliefs. In the "upper dominance region", the signal is so high that a late consumer always prefers to wait until date 2 to withdraw. Following Goldstein and Pauzner (2005), we assume that this happens above a threshold $\bar{\sigma}$, where the productive asset is safe, i.e. $p = 1$, and gives the same return R at date 1 and 2. In this way, a late consumer who waits is sure to get $(R(1-L)+RL-nc)/(1-n) > c$ at date 2, hence he or she will not run for any fraction n of depositors withdrawing early. In the "lower dominance region", instead, the signal is so low that a late consumer always runs, irrespective of the behavior of the other depositors, thus triggering a "fundamental run". This happens below the threshold signal $\underline{\sigma}_j$, that makes him or her indifferent between withdrawing or not, and depends on the pecking order j chosen by the bank (we characterize the thresholds in the incoming sections).

The existence of the lower and upper dominance regions, regardless of their size, ensures the existence of an equilibrium in the intermediate region $[\underline{\sigma}_j, \bar{\sigma}]$, where the late consumers decide whether to run or not based on a threshold strategy: They run if the signal is lower than a threshold signal σ_j^*.[12] Let $Prob(\sigma \leq \sigma_j^*)$ be the probability that $\sigma \leq \sigma_j^*$ under pecking order j. Then, given $\sigma = p + e$, we have:

$$Prob(\sigma \leq \sigma_j^*) = \int_{-\epsilon}^{\sigma_j^* - p} \frac{1}{2\epsilon} de = \max\left(\frac{\sigma_j^* - p + \epsilon}{2\epsilon}, 0\right). \tag{13}$$

Define as $c_L(Z, n)$ the amount of late consumption that a late consumer would get if the realized aggregate productivity shock is Z and n depositors withdraw at date 1. Arguably, it should be the case that the higher the fraction of depositors who run is, the lower late consumption is, or $\partial c_L(Z, n)/\partial n \leq 0$. Moreover, define n_j^{**} as the maximum fraction of depositors that a bank can serve under pecking order j without breaking the deposit contract, i.e. while still being solvent and able to pay c to all those depositors who withdraw at date 1. For $n \geq n_j^{**}$, the bank goes into bankruptcy: There are no more resources for late consumption, the bank pays $c^B(n)$ according to an equal service constraint, i.e. it equally splits the total liquidation value of its asset portfolio among the n depositors who withdraw at date 1, and then closes down.

Define the expected utility from waiting $\mathbb{E}[u(c_L(Z, n))]$ given the signal σ and the fraction n of depositors who withdraw at date 1 as:

$$\mathbb{E}[u(c_L(Z, n))] = \int_{-\epsilon}^{\epsilon} (\sigma - e) u(c_L(R, n)) \frac{1}{2\epsilon} de + \int_{-\epsilon}^{\epsilon} (1 - \sigma + e) u(c_L(0, n)) \frac{1}{2\epsilon} de. \tag{14}$$

[12] In the present environment, Goldstein and Pauzner (2005) prove that the equilibrium strategy is always a threshold strategy.

It is immediate to verify that this reduces to:

$$\mathbb{E}[u(c_L(Z,n))] = \sigma u(c_L(R,n)) + (1-\sigma)u(c_L(0,n)). \tag{15}$$

Then, the utility advantage of waiting versus running under pecking order j, for a given fraction n of depositors who withdraw at date 1, is:

$$v_j(n) = \begin{cases} \sigma u(c_L(R,n)) + (1-\sigma)u(c_L(0,n)) - u(c) & \text{if } \lambda \leq n < n_j^{**}, \\ -u(c(n)) & \text{if } n_j^{**} \leq n < 1. \end{cases} \tag{16}$$

The fraction of depositors who withdraw at date 1 is given by the sum of the λ early consumers and the $1-\lambda$ late consumers who receive a signal lower than the threshold signal σ_j^*:

$$n = \lambda + (1-\lambda)Prob(\sigma \leq \sigma_j^*) = \lambda + (1-\lambda)\max\left(\frac{\sigma_j^* - p + \epsilon}{2\epsilon}, 0\right). \tag{17}$$

Thus, n is a random variable that depends on the aggregate state of the economy. Importantly, as σ is a random variable, its cumulative distribution function $Prob(\sigma \leq \sigma_j^*)$ is uniformly distributed over the interval $[0,1]$ by the Laplacian Property (Morris and Shin, 1998). Thus, the fraction of depositors n who withdraw at date 1 must also be uniformly distributed, over the interval $[\lambda, 1]$. This allows us to calculate the expected value of waiting versus running as:

$$\mathbb{E}[v_j(n)|\sigma] = \int_\lambda^1 \frac{v_j(n)}{1-\lambda} dn, \tag{18}$$

and to characterize the threshold signal σ_j^* as the one such that $\mathbb{E}[v_j(n)|\sigma_j^*] = 0$.

From what said so far, it is clear that the decision of a late consumer about whether to run depends on the decision of the bank about how to pay early withdrawals, i.e. on the pecking order with which it employs liquidation of the productive asset and liquidity. In what follows, we characterize and compare the withdrawal behavior of the depositors under each pecking order, by studying its effects on the lower dominance region and the threshold strategies.

3.1 Pecking order 1: {Liquidation; Liquidity}

In this first case, the bank serves the depositors who withdraw at date 1 first by liquidating the productive asset, and then by deploying the liquidity in portfolio. Under this pecking order, the threshold signal $\underline{\sigma}_1$ characterizing the lower dominance region is the one that equalizes:

$$u(c) = \underline{\sigma}_1 u\left(\frac{R\left(1 - L - \frac{\lambda c}{r}\right) + L}{1-\lambda}\right) + (1-\underline{\sigma}_1)u\left(\frac{L}{1-\lambda}\right). \tag{19}$$

This expression states that a late consumer receiving a signal $\underline{\sigma}_1$ must be indifferent between withdrawing at date 1 and getting c and waiting until date 2 and getting $c_L(R,\lambda)$ with probability $\underline{\sigma}_1$ or $c_L(0,\lambda)$ with probability $1-\underline{\sigma}_1$. These values come from the fact that, by liquidating the productive asset first, the bank withholds liquidity, that pays late consumption irrespective of the realization of the aggregate productivity shock Z. Moreover, the bank has to pay an amount of early consumption c to λ early consumers, by liquidating an amount D of productive assets at rate r, hence $D = \lambda c/r$. Rearranging the equality above, we obtain the threshold:

$$\underline{\sigma}_1 = \frac{u(c) - u\left(\frac{L}{1-\lambda}\right)}{u\left(\frac{R\left(1-L-\frac{\lambda c}{r}\right)+L}{1-\lambda}\right) - u\left(\frac{L}{1-\lambda}\right)}, \tag{20}$$

which is clearly increasing in the amount of early consumption c set in the deposit contract.

The threshold strategy in the intermediate region $[\underline{\sigma}_1, \bar{\sigma}]$ instead depends on the late consumers' advantage of waiting versus running:

$$v_1(n) = \begin{cases} \sigma u\left(\frac{R\left(1-L-\frac{nc}{r}\right)+L}{1-n}\right) + (1-\sigma)u\left(\frac{L}{1-n}\right) - u(c) & \text{if } \lambda \leq n < n_1^*, \\ \sigma u\left(\frac{r(1-L)+L-nc}{1-n}\right) + (1-\sigma)u\left(\frac{r(1-L)+L-nc}{1-n}\right) - u(c) & \text{if } n_1^* \leq n < n_1^{**}, \\ -u\left(\frac{r(1-L)+L}{n}\right) & \text{if } n_1^{**} \leq n < 1. \end{cases} \tag{21}$$

In this expression, $n_1^* = (r(1-L))/c$ and $n_1^{**} = (r(1-L)+L)/c$ are the maximum fractions of depositors that a bank can serve at date 1 without breaking the deposit contract, and either liquidating the whole amount of productive assets in portfolio (up to n_1^*) or using also liquidity (up to n_1^{**}). When the fraction of depositors who withdraw at date 1 lies in the interval $[\lambda, n_1^*]$, the bank fulfills its contractual obligation by liquidating the productive asset first: It needs to pay an amount of early consumption c to n depositors via rD resources from liquidation, hence the amount of productive asset to liquidate is $D = nc/r$. Then, if n depositors withdraw at date 1, the consumption of a late consumer who waits until date 2 is:

$$c_L(Z, n) = \frac{Z\left(1 - L - \frac{nc}{r}\right) + L}{1 - n}, \tag{22}$$

depending on the realization of the aggregate productivity shock Z. When the fraction of depositors who withdraw at date 1 lies in the interval $[n_1^*, n_1^{**}]$, the bank instead fulfills its contractual obligation by liquidating all productive assets in portfolio (thus generating resources equal to $r(1 - L)$) and by deploying liquidity. Thus, if n depositors withdraw at date 1, the consumption of a late consumer who waits until date 2 is independent of the realization of the aggregate productivity shock Z (as the productive assets have all been liquidated) and equal to $c_L^L(n) = (r(1-L)+L-nc)/(1-n)$. Finally, when the fraction of depositors who withdraw at date 1 lies in the interval $[n_1^{**}, 1]$, the bank goes bankrupt, as it does not hold sufficient resources to pay an amount of early consumption c to all depositors. In this case, the bank is forced to liquidate all productive assets and close down, so a late consumer who waits until date 2 gets zero. Moreover, the available resources (equal to $r(1 - L) + L$) are equally split among all the n depositors who withdraw at date 1, and each one gets $c^B(n) = (r(1 - L) + L)/n$.

Figure 4 shows the evolution of liquidity holdings under this pecking order. When $n = \lambda$, the bank holds an amount of liquidity L from date 0, and creates further liquidity by liquidating the productive asset to pay λc total early withdrawals. In the interval $[\lambda, n_1^*]$, the bank engages in liquidity hoarding, i.e. it retains the liquidity in its portfolio and accumulates more of it by liquidating the productive asset, up to the point at n_1^* where it has liquidated all the productive asset in portfolio and generated the maximum amount of liquidity $L + r(1 - L)$. Finally, in the interval $[n_1^*, n_1^{**}]$ the bank has no more holdings of the productive asset, and start depleting its liquidity holdings to pay early withdrawals, up to the point of bankruptcy at n_1^{**}.

The sign of the strategic complementarity affecting the decision of a late consumer to run depends on how the advantage of waiting versus running varies with the fraction of depositors

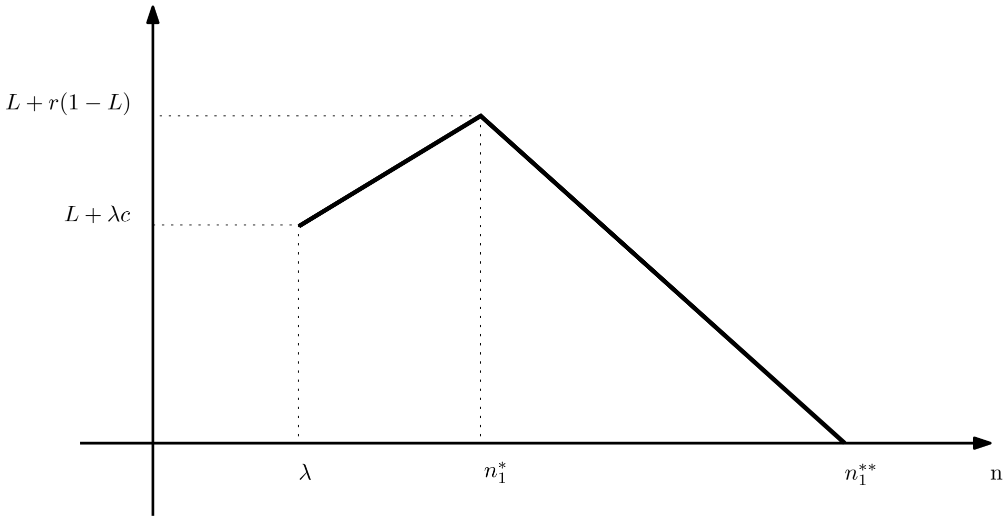

Figure 4: Bank liquidity holdings during a run under the pecking order {Liquidation; Liquidity}.

withdrawing at date 1. More formally:

$$\frac{\partial v_1}{\partial n} = \begin{cases} \sigma u'(c_L(R,n)) \frac{-\frac{R}{r}c(1-n)+\left[R\left(1-L-\frac{nc}{r}\right)+L\right]}{(1-n)^2} + \frac{(1-\sigma)u'(c_L(0,n))L}{(1-n)^2} & \text{if } \lambda \leq n < n_1^*, \\ u'(c_L^L(n)) \frac{r(1-L)+L-c}{(1-n)^2} & \text{if } n_1^* \leq n < n_1^{**}, \\ u'(c^B(n)) \frac{c^B(n)}{n} & \text{if } n_1^{**} \leq n < 1. \end{cases} \quad (23)$$

On the one side, in the interval $[n_1^{**}, 1]$ the derivative is positive, as after bankruptcy equal service prescribes total resources to be shared pro-rata to all depositors; on the other side, in the interval $[n_1^*, n_1^{**}]$ the derivative is negative by definition of n_1^{**}, highlighting the presence of one-sided strategic complementarities. We characterize the direction of the strategic complementarity in the interval $[\lambda, n_1^*]$ in the following Lemma:

Lemma 3. *In the interval* $[\lambda, n_1^*]$, $v_1(n)$ *is decreasing in* n.

Proof. In Appendix A. ■

Figure 5 shows that under the pecking order {Liquidation; Liquidity} the economy exhibits one sided strategic complementarities: The advantage of waiting versus running is decreasing in the fraction of depositors running before bankruptcy, and increasing after bankruptcy. However, despite not knowing the sign of $v_1(n_1^*)$, the function $v_1(n)$ crosses zero only once, because is decreasing in n in both intervals $[\lambda, n_1^*]$ and $[n_1^*, n^*]$. Moreover, the advantage of waiting versus running is increasing

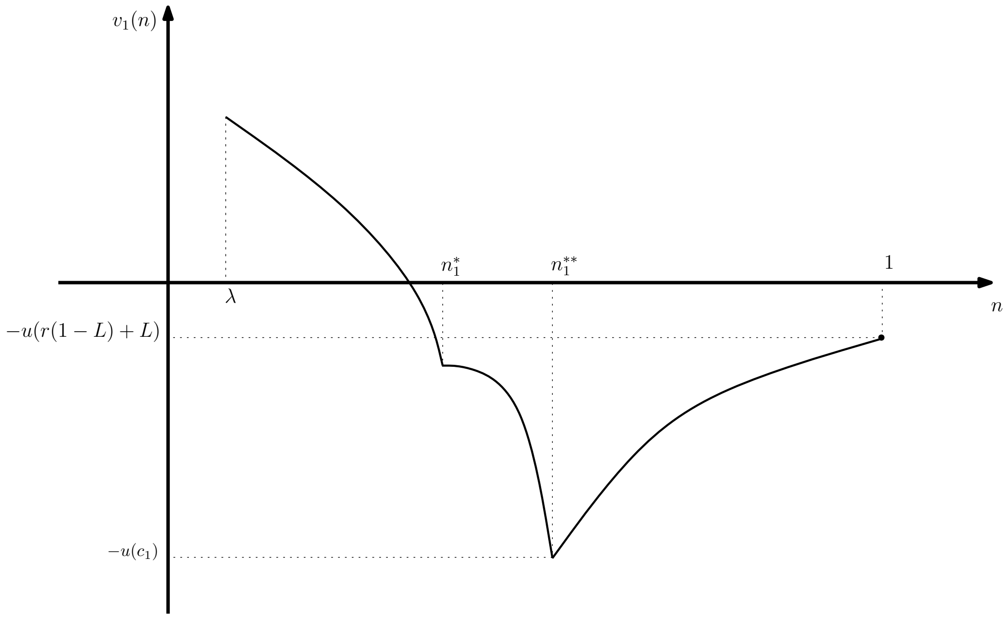

Figure 5: The advantage of waiting versus running, as a function of the fraction of depositors running, when the bank chooses the pecking order {Liquidation; Liquidity}.

in σ in the interval $[\lambda, n_1^*]$ as $c_L(R,n) \geq c_L(0,n)$, and is independent of σ in the interval $[n_1^*, n^{**}]$. Together, these properties guarantee the uniqueness of the equilibrium in the intermediate region $[\underline{\sigma}_1, \bar{\sigma}]$ (Goldstein and Pauzner, 2005).

Lemma 4. *Under the pecking order {Liquidation; Liquidity}, in the intermediate region* $[\underline{\sigma}_1, \bar{\sigma}]$ *a late consumer runs if his or her signal is lower than the threshold signal:*

$$\sigma_1^* = \frac{\int_\lambda^{n_1^{**}} u(c)dn + \int_{n_1^{**}}^1 u\left(\frac{L+r(1-L)}{n}\right)dn - \int_\lambda^{n_1^*} u\left(\frac{L}{1-n}\right)dn - \int_{n_1^*}^{n_1^{**}} u\left(\frac{L+r(1-L)-nc}{1-n}\right)dn}{\int_\lambda^{n_1^*}\left[u\left(\frac{R\left(1-L-\frac{nc}{r}\right)+L}{1-n}\right) - u\left(\frac{L}{1-n}\right)\right]dn}. \tag{24}$$

The threshold signal σ_1^* *is increasing in* c *and decreasing in* L.

Proof. In Appendix A. ■

The Lemma characterizes the endogenous threshold signal below which all late consumers run, and the effects that the bank's deposit contract and asset portfolio have on it. In particular, increasing early consumption c has a threefold positive effect on the threshold signal σ_1^*: It directly increases the advantages for a late consumer to run, both before and after bankruptcy; it lowers the maximum fraction of depositors that a bank can serve before bankruptcy; it decreases the advantages of waiting until date 2. The effect that increasing the total amount of liquidity in the bank's portfolio has on the threshold signal σ_1^* instead looks ambiguous. However, the effect that one more unit of liquidity has on the marginal utility of those late consumers not running just before bankruptcy (i.e. as n approaches n_1^{**} in the interval $[n_1^*, n_1^{**}]$ in the numerator of σ_1^*) dominates: More liquidity allows them to consume a positive amount instead of zero, and this has a big effect on their marginal utility. Thus, the threshold signal σ_1^* turns out to be decreasing in L.

3.2 Pecking order 2: {Liquidity; Liquidation}

In this second case, we assume that the bank serves the depositors who withdraw at date 1 first by deploying liquidity, and then by liquidating the productive asset. Under this pecking order, the threshold signal $\underline{\sigma}_2$ characterizing the lower dominance region is the one that equalizes:

$$u(c) = \underline{\sigma}_2 u\left(\frac{R(1-L)+L-\lambda c}{1-\lambda}\right) + (1-\underline{\sigma}_2)u\left(\frac{L-\lambda c}{1-\lambda}\right). \tag{25}$$

This expression states that a late consumer receiving a signal $\underline{\sigma}_2$ must be indifferent between withdrawing at date 1 and getting c and waiting until date 2 and getting $c_L(R,\lambda) = (R(1-L) + L - \lambda c)/(1-\lambda)$ with probability $\underline{\sigma}_2$ or $c_L(0,\lambda) = (L-\lambda c)/(1-\lambda)$ with probability $1-\underline{\sigma}_2$. These values come from the fact that, by deploying liquidity first, the bank withholds the productive asset. Hence, having to pay an amount of early consumption c to λ early consumers, it rolls over an amount $L-\lambda c$ of precautionary liquidity from date 1 to date 2. Rearranging the equality above, we obtain the threshold:

$$\underline{\sigma}_2 = \frac{u(c) - u\left(\frac{L-\lambda c}{1-\lambda}\right)}{u\left(\frac{R(1-L)+L-\lambda c}{1-\lambda}\right) - u\left(\frac{L-\lambda c}{1-\lambda}\right)}. \tag{26}$$

As for the pecking order {Liquidation; Liquidity}, this threshold is increasing in the amount of early consumption c set in the deposit contract. To see that, it suffices to calculate:

$$\frac{\partial \underline{\sigma}_2}{\partial c} = \frac{u'(c) + \frac{\lambda}{1-\lambda}u'\left(\frac{L-\lambda c}{1-\lambda}\right) + \underline{\sigma}_2 \frac{\lambda}{1-\lambda}\left[u'\left(\frac{R(1-L)+L-\lambda c}{1-\lambda}\right) - u'\left(\frac{L-\lambda c}{1-\lambda}\right)\right]}{u\left(\frac{R(1-L)+L-\lambda c}{1-\lambda}\right) - u\left(\frac{L-\lambda c}{1-\lambda}\right)}, \tag{27}$$

and notice that it is always positive, as $\underline{\sigma}_2$ is lower than 1.

The threshold strategy in the intermediate region $[\underline{\sigma}_2, \bar{\sigma}]$ instead depends on the late consumers' advantage of waiting versus running:

$$v_2(n) = \begin{cases} \sigma u\left(\frac{R(1-L)+L-nc}{1-n}\right) + (1-\sigma)u\left(\frac{L-nc}{1-n}\right) - u(c) & \text{if } \lambda \le n < n_2^*, \\ \sigma u\left(\frac{R(1-L-D)}{1-n}\right) - u(c) = \sigma u\left(\frac{R\left(1-L-\frac{nc-L}{r}\right)}{1-n}\right) - u(c) & \text{if } n_2^* \le n < n_2^{**}, \\ -u\left(\frac{L+r(1-L)}{n}\right) & \text{if } n_2^{**} \le n < 1. \end{cases} \tag{28}$$

Similarly to the previous case, $n_2^* = L/c$ and $n_2^{**} = (r(1-L)+L)/c$ are the maximum fractions of depositors that a bank can serve at date 1 without breaking the deposit contract and using liquidity (up to n_2^*), and also liquidating the whole amount of productive assets in portfolio (up to n_2^{**}). When the fraction of depositors who withdraw at date 1 lies in the interval $[\lambda, n_2^*]$, the bank fulfills its contractual obligation by keeping the productive asset and using liquidity. Hence, if n depositors are withdrawing at date 1, the consumption of a late consumer who waits until date 2 is either $c_L(R,n) = (R(1-L)+L-nc)/(1-n)$ or $c_L(0,n) = (L-nc)/(1-n)$, depending on the realization of the aggregate productivity shock Z. When the fraction of depositors who withdraw at date 1 lies instead in the interval $[n_2^*, n_2^{**}]$, the bank is forced to fulfill its contractual obligation also by liquidating the productive assets in portfolio. Hence, the total available resources to provide early consumption c to the n depositors who withdraw at date 1 are $L + rD$, and the amount that the bank liquidates is equal to $D = \frac{nc-L}{r}$. Moreover, as the liquidity has been exhausted, the consumption of a late consumer who waits until date 2 and finds herself in the state where the aggregate productivity shock Z is zero, while when Z is positive is:

$$c_L^D(R,n) = \frac{R\left(1-L-\frac{nc-L}{r}\right)}{1-n}. \tag{29}$$

Finally, when the fraction of depositors who withdraw at date 1 lies in the interval $[n_2^{**}, 1]$, the bank is bankrupt. Thus, by the equal service constraint, all the n depositors who withdraw at date 1 get $c^B(n) = (r(1-L)+L)/n$, and those $1-n$ who do not withdraw get zero.

Figure 6 shows the evolution of liquidity holdings under this pecking order. When $n = \lambda$, the bank holds an amount of liquidity L from date 0, and employs it to pay λc total early withdrawals. In the interval $[\lambda, n_2^*]$, the bank engages in liquidity cushioning, i.e. it depletes the liquidity in its portfolio, up to the point at n_2^* where it has completely run out of it. Finally, in the interval

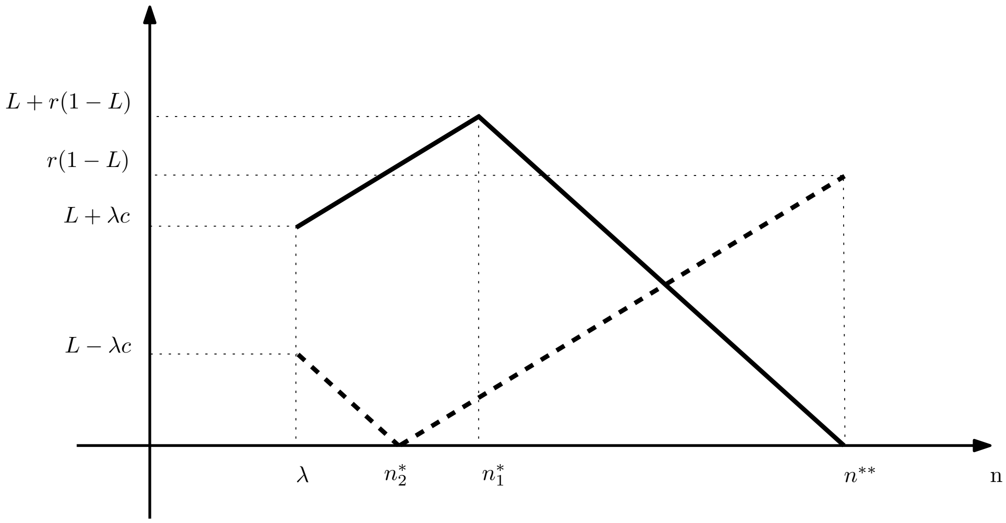

Figure 6: Bank liquidity holdings during a run under the pecking order {Liquidation; Liquidity} (solid line) and {Liquidity; Liquidation} (dashed line).

$[n_2^*, n_2^{**}]$ the bank starts creating new liquidity by liquidating the productive asset, up to the point of bankruptcy at n_2^{**} where the maximum amount of liquidity generated is $r(1-L)$. Notice that the total fraction of depositors that can be served before bankruptcy is the same under the two pecking orders. Hence, to economize on notation, we write $n_1^{**} = n_2^{**} = n^{**}$.

We again study the sign of the strategic complementarities by taking the derivative of $v_2(n)$ with respect to n:

$$\frac{\partial v_2}{\partial n} = \begin{cases} \sigma u'(c_L(R,n))\frac{c_L(R,n)-c}{1-n} - (1-\sigma)u'(c_L(0,n))\frac{c-c_L(0,n)}{1-n} & \text{if } \lambda \leq n < n_2^*, \\ \sigma u'(c_L^D(R,n))\frac{c_L^D(R,n)-\frac{Rc}{r}}{1-n} & \text{if } n_2^* \leq n < n^{**}, \\ u'(c^B(n))\frac{c^B(n)}{n} & \text{if } n^{**} \leq n < 1. \end{cases} \tag{30}$$

As before, in the interval $[n^{**}, 1]$ the derivative is positive, while in the interval $[n_2^*, n^{**}]$ is negative by definition of n^{**}. We characterize the sign of the strategic complementarity in the interval $[\lambda, n_2^*]$ in the following Lemma:

Lemma 5. *In the interval $[\lambda, n_2^*]$, $v_2(n)$ is decreasing in n whenever is non-positive.*

Proof. In Appendix A. ■

In order to guarantee the uniqueness of the equilibrium, we first need to show that $v(n_2^*) < 0$.

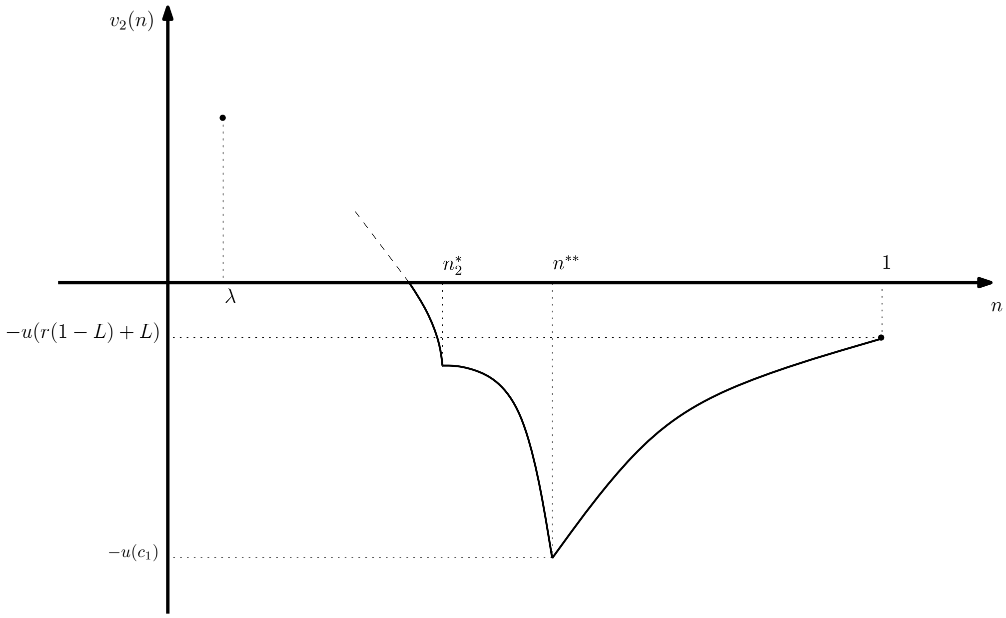

Figure 7: The advantage of waiting versus running, as a function of the fraction of depositors running, when the bank chooses the pecking order {Liquidity; Liquidation}.

To this end, notice that:

$$v_2(n_2^*) = \sigma u\left(\frac{R(1-L)}{c-L}c\right) + (1-\sigma)u(0) - u(c). \tag{31}$$

This expression is negative if:

$$\sigma < \frac{u(c)}{u\left(\frac{R(1-L)}{c-L}c\right)} \equiv \tilde{\sigma}, \tag{32}$$

where $\tilde{\sigma} > 1$ whenever $R < (c-L)/(1-L)$. In the proof of Proposition 2, we show that this condition holds in the banking equilibrium under the pecking order {Liquidity, Liquidation}. Hence, $v_2(n_2^*) < 0$, because σ is always lower than 1 by definition. Moreover, as in the previous case the advantage of waiting versus running is increasing in σ in the interval $[\lambda, n_2^*]$ as $c_L(R,n) \geq c_L(0,n)$, and is clearly also increasing in σ in the interval $[n_2^*, n^{**}]$. These properties guarantee that the function $v_2(n)$ crosses zero only once in the interval $[\lambda, n^{**}]$, and that is sufficient for a solution to exist and be unique (Goldstein and Pauzner, 2005).

With this result in hand, we characterize the threshold signal that makes a late consumer indifferent between waiting or running under the pecking order {Liquidity; Liquidation}:

Lemma 6. *Under the pecking order {Liquidity; Liquidation}, in the intermediate region* $[\underline{\sigma}_2, \bar{\sigma}]$ *a* *late consumer runs if his or her signal is lower than the threshold signal:*

$$\sigma_2^* = \frac{\displaystyle\int_{\lambda}^{n^{**}} u(c)dn + \int_{n^{**}}^{1} u\left(\tfrac{L+r(1-L)}{n}\right) dn - \int_{\lambda}^{n_2^*} u\left(\tfrac{L-nc}{1-n}\right) dn}{\displaystyle\int_{\lambda}^{n_2^*} \left[u\left(\tfrac{R(1-L)+L-nc}{1-n}\right) - u\left(\tfrac{L-nc}{1-n}\right)\right] dn + \int_{n_2^*}^{n^{**}} u\left(\tfrac{R\left(1-L-\frac{nc-L}{r}\right)}{1-n}\right) dn}. \tag{33}$$

The threshold signal σ_2^* is increasing in c, and decreasing in L.

Proof. In Appendix A. ■

Intuitively, the Lemma shows that increasing early consumption c has a positive effect on the threshold signal σ_2^* for many concurrent reasons. First, as in the pecking order {Liquidation, Liquidity}, early consumption directly increases the advantages of running before bankruptcy. Moreover, it decreases the advantages of waiting until date 2, either by decreasing the amount of precautionary liquidity $L - \lambda c$ rolled over to date 2 or by forcing the bank to liquidate more productive assets, whenever the liquidity has been completely exhausted. Finally, increasing c has a negative effect on the amount of insurance that a bank can provide against the aggregate productivity shock Z, and that in turns increases the threshold signal and the incentives to run. In contrast, increasing the amount of liquidity has an ambiguous effect on the threshold signal. However, the effect that one more unit of liquidity has on the marginal utility of those late consumers not running (i) in the bad state of the world just before the bank runs out of liquidity (i.e. at n_2^* in the interval $[\lambda, n_2^*]$ in the numerator of σ_2^*) and (ii) just before bankruptcy (i.e. as n approaches n^{**} in the interval $[n_2^*, n^{**}]$ in the denominator of σ_2^*) is again large. Thus, the threshold probability σ_2^* is decreasing in L.

4 Banking equilibrium

The previous section characterized the depositors' withdrawal behavior under the two pecking orders. With this in hand, we now solve for the banking equilibrium of Definition 1. At date 0, the bank chooses first the deposit contract and asset portfolio, and then the endogenous pecking order with which to use the available assets to pay withdrawals at date 1. By backward induction, we start from the latter.

4.1 Endogenous pecking order

For given deposit contract and the asset portfolio, the bank decides the optimal pecking order as a best response to the withdrawal decisions of the depositors. Importantly, as we are solving for a Perfect Bayesian Equilibrium, it is necessary to define the bank's strategy both on and off the equilibrium path. This is crucial, because the pecking-order strategy defines the endogenous threshold signal σ_j^*, and as a consequence the deposit contract and asset portfolio that the bank is going to choose on equilibrium.

On the equilibrium path, because we focus on a symmetric equilibrium, only two outcomes are possible: Either no depositor runs and the bank is solvent, or all depositors run and the bank is insolvent. In the first case, the outcome is equivalent to the one with perfect information of section 2.2, where the bank never liquidates the productive asset, and therefore the pecking order is immaterial. In the second case, the bank is instead forced to liquidate all the productive assets in portfolio, so the pecking order is again immaterial. Therefore, to complete the characterization of the optimal pecking-order strategy, we only need to focus on the off-equilibrium case in which the fraction n of depositors running is below n^{**} and the bank is illiquid but solvent.

Recalling the timing of actions, notice that a bank solving the dual problem maximizes:

$$\int_0^{\sigma_j^*} u(L + r(1-L))dp + \int_{\sigma_j^*}^1 \Big[\lambda u(c) + (1-\lambda)\Big[pu(c_L(R)) + (1-p)u(c_L(0))\Big]\Big] dp. \tag{34}$$

This is the expected utility of the depositors, when the bank offers an amount c of early consumption, holds an amount L of liquidity, and chooses the pecking order j inducing the threshold signal σ_j^*. If $c \geq L + r(1-L)$ and $L < 1$, the above expression is decreasing in σ_j^* for any possible value of c and L. Hence, the expected utility is maximized at $\sigma^{BE} \equiv \text{argmin}\ \{\sigma_1^*, \sigma_2^*\}$. The equilibrium threshold signal σ^{BE} depends on the recovery rate from liquidating the productive asset and on the depositors' relative risk aversion. The following Proposition characterizes the equilibrium pecking order off the equilibrium path, and summarizes the argument on the equilibrium path:

Proposition 1. *Off the equilibrium path, where the bank is illiquid but solvent, if the coefficient of relative risk aversion is sufficiently high, there exists a unique threshold* $\tilde{r} \in [0, 1]$ *such that for any* $r \leq \tilde{r}$ *the optimal pecking order is {Liquidity; Liquidation} and* $\sigma^{BE} = \sigma_2^*$*, and for any* $r > \tilde{r}$ *is*

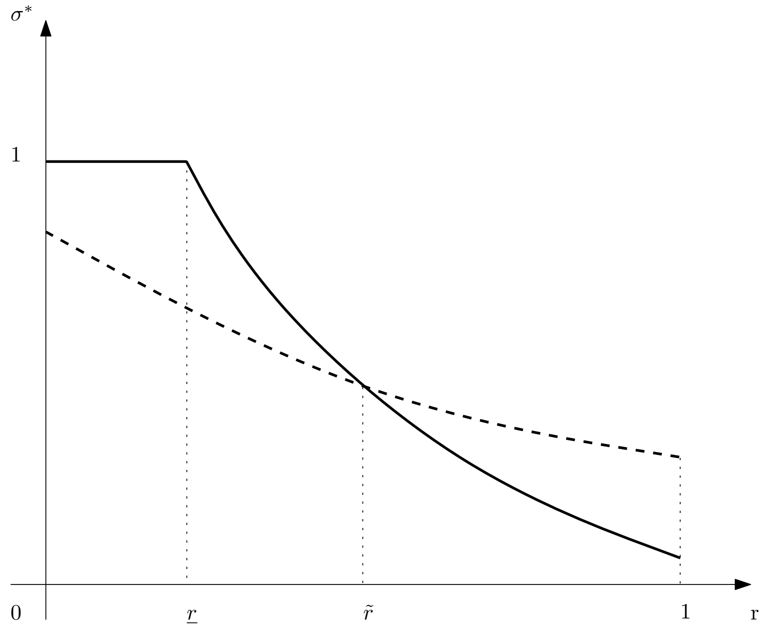

Figure 8: The threshold signals under the pecking order {Liquidation; Liquidity} (solid line) and {Liquidity; Liquidation} (dashed line) for different values of the recovery rate of the productive asset (on the x-axis).

{Liquidation; Liquidity} and $\sigma^{BE} = \sigma_1^*$*. On the equilibrium path, where the bank is either solvent or insolvent, the choice of the optimal pecking order is immaterial.*

Proof. In Appendix A. ■

The proof of the first part of the Proposition is based on showing that the threshold signals under the two pecking orders adjust to changes in the recovery rate of the productive asset as Figure 8 shows. First, both threshold signals σ_1^* and σ_2^* are decreasing and convex functions of the recovery rate r. This happens because, when the fraction of depositors who are running is n^{**} (i.e. the value that triggers bankruptcy under both pecking orders) a late consumer who does not join a run gets zero. Hence, increasing the recovery rate by one marginal unit makes his or her consumption go from zero to a positive value. This by the Inada conditions has a large positive effect on the utility of waiting (although decreasing because of the concavity of $u(c)$) and lowers both threshold signals in a convex way.

Second, the comparison between the two pecking orders essentially boils down to comparing the costs associated with using either liquidation or liquidity to pay early withdrawals. On the one hand, liquidation of the productive asset at date 1 is costly in terms of (i) forgone resources due to the deadweight losses from liquidation (as $r < 1$) and (ii) forgone late consumption in the good state of the world. On the other hand, using liquidity is costly in terms of forgone late consumption in the bad state of the world, i.e. in terms of lower insurance against the aggregate productivity shock. If the depositors are sufficiently risk averse and the recovery rate r is close to 1, both costs associated with liquidation become less relevant, because the depositors care relatively less about high late consumption in the good state of the word and the bank wastes less resources when liquidating the productive asset. The opposite is true with respect to the cost associated with using liquidity because, being very risk averse, the depositors care a lot about late consumption in the bad state of the world. Therefore, with sufficiently high relative risk aversion and a recovery rate r close to 1, {Liquidation; Liquidity} is the optimal pecking order and the equilibrium threshold signal is $\sigma^{BE} = \sigma_1^*$.

If instead the recovery rate is close to zero, liquidation becomes very costly, and this is enough to ensure that {Liquidity; Liquidation} is the optimal pecking order and the equilibrium threshold signal is $\sigma^{BE} = \sigma_2^*$. This happens because a late consumer who does not join a run is worse off under the pecking order {Liquidation; Liquidity} than under {Liquidity; Liquidation}. On the one side, the threshold signal σ_1^* under the pecking order {Liquidation; Liquidity} is constant and equal to one, i.e. there exists a lower bound $\underline{r}$ for the recovery rate, below which all late consumers would rather withdraw at date 1 than at date 2, irrespective of the fraction of depositors running, hence any signal would lead to a run. On the other side, the threshold signal σ_2^* under the pecking order {Liquidity; Liquidation} is always lower than 1 when the recovery rate is equal to zero.

To sum up, under the assumption of Proposition 1, the graphs of the two threshold signals meet at most once for any recovery rate in the interval $[0, 1]$. This means that off the equilibrium path the bank prefers the pecking order {Liquidation; Liquidity} only if the recovery rate of the productive asset is sufficiently high, so that it can liquidate at lower costs and roll over liquidity to the final period to ensure the depositors against the aggregate productivity shock Z. If instead the recovery rate of the productive asset is low, the bank prefers the pecking order {Liquidity; Liquidation}.

4.2 Equilibrium deposit contract and asset portfolio

For given recovery rate r, Proposition 1 characterizes the equilibrium pecking order, and the corresponding equilibrium threshold σ^{BE}. Then, the bank finds the optimal deposit contract and asset portfolio by solving:

$$\max_{c,L} \int_0^{\sigma^{BE}} u(L + r(1-L))dp + \int_{\sigma^{BE}}^1 \left[\lambda u(c) + (1-\lambda)\left[pu\left(\frac{R(1-L)+L-\lambda c}{1-\lambda}\right) + \right.\right.$$
$$\left.\left. + (1-p)u\left(\frac{L-\lambda c}{1-\lambda}\right)\right]\right] dp, \tag{35}$$

subject to the liquidity constraint $L \geq \lambda c$. When the signal is below the threshold signal σ^{BE} a run happens, either fundamental or self-fulfilling: All depositors get an equal share of the liquidation value of the whole asset portfolio, equal to $L + r(1-L)$. When instead the signal is above the threshold signal σ^{BE}, a run does not happen: A fraction λ of depositors are early consumers, and consume c, while a fraction $1-\lambda$ of them are late consumers, and consume either $c_L(R) = (R(1-L)+L-\lambda c)/(1-\lambda)$ if the productive assets yields a positive return, or $c_L(0) = (L-\lambda c)/(1-\lambda)$ if it yields zero. Define the difference between the utility in the case of no-run and the utility in the case of run as:

$$\Delta U(c, L) = \lambda u(c) + (1-\lambda)\left[\sigma^{BE} u\left(\frac{R(1-L)+L-\lambda c}{1-\lambda}\right) + (1-\sigma^{BE})u\left(\frac{L-\lambda c}{1-\lambda}\right)\right] +$$
$$- u(L + r(1-L)). \tag{36}$$

Then, from the first-order conditions of the program, we prove the following:

Proposition 2. *The banking equilibrium features precautionary liquidity ($L^{BE} > \lambda c^{BE}$). The equilibrium deposit contract and asset portfolio satisfy the distorted Euler equation:*

$$\int_{\sigma^{BE}}^1 \left[u'(c^{BE}) - pRu'(c_L^{BE}(R))\right] dp + \sigma^{BE}(1-r)u'(L^{BE} + r(1-L^{BE})) =$$
$$= \left[\frac{\partial \sigma^{BE}}{\partial L} + \frac{1}{\lambda}\frac{\partial \sigma^{BE}}{\partial c}\right] \Delta U(c^{BE}, L^{BE}). \tag{37}$$

Moreover, the equilibrium deposit contract satisfies:

$$0 < c_L^{BE}(0) < c^{BE} < c_L^{BE}(R). \tag{38}$$

Proof. In Appendix A. ■

The proof of the Proposition is in part similar to the one of the equilibrium with perfect information. By the Inada conditions, the bank finds optimal to let late consumers avoid zero consumption in the bad state of the world. Hence, it provides insurance against the aggregate productivity shock by engaging in precautionary savings, i.e. by holding more liquidity than the one needed to insure the depositors against the idiosyncratic liquidity shocks. However, differently from the equilibrium with perfect information, the bank further imposes a wedge between the marginal rate of substitution between early and late consumption and the expected marginal rate of transformation of the productive asset (the first term on the left-hand side of (37)). This happens through two channels. First, the bank takes into account that it needs higher liquidity to pay consumption in the case of a run (the second term on the left-hand side of (37)). Second, it also takes into account that the equilibrium deposit contract and asset portfolio affect the equilibrium threshold signal σ^{BE}, and therefore the probability that a run is realized (the right-hand side of (37)). The effect of the wedge is to distort the equilibrium allocation with respect to the equilibrium with perfect information. The direction of this distortion depends on the sign of the wedge and on the probability of the idiosyncratic liquidity shock:

Corollary 1. *If λ is sufficiently large, the banking equilibrium features extra precautionary liquidity: $c^{BE} < c^{PI}$ and $L^{BE} > L^{PI}$, hence precautionary liquidity is higher than in the equilibrium with perfect information.*

Proof. In Appendix A. ■

Intuitively, the anticipation of self-fulfilling runs imposes a positive wedge between the marginal rate of substitution between early and late consumption and the expected marginal rate of transformation of the productive asset. The wedge forces the bank to lower early consumption and increase its liquidity holding with respect to the equilibrium with perfect information. In other words, the bank reacts to the anticipation of self-fulfilling runs by further increasing the amount of precautionary liquidity above the one needed to insure the depositors against the aggregate productivity shock. This happens because the marginal effect of the bank asset portfolio on the equilibrium threshold signal σ^{BE} (the right-hand side of the distorted Euler equation (37)) is larger than the expected marginal utility of consumption at a run (the second term on the left-hand side of (37)). A sufficient condition for this to happen is that the utility differential $\Delta U(c, L)$ between no-run and run is sufficiently large, which in turns is guaranteed if the probability of the idiosyncratic liquidity shock λ is sufficiently large. Moreover, the marginal effect of one additional unit of liquidity on the equilibrium threshold signal σ^{BE} is always negative, and is stronger under {Liquidation; Liquidity} than under {Liquidity; Liquidation}, because under the first pecking order liquidity is retained in the initial stages of a run and consumed at date 2 in the bad state of the world, and as a consequence has a larger diminishing effect on the depositors' incentives to run. Thus, in order to induce a positive wedge and extra precautionary liquidity, the requirement on the utility differential $\Delta U(c, L)$ (and consequently on λ) is stronger under the pecking order {Liquidation; Liquidity} than under {Liquidity; Liquidation}.

In sum, the banking equilibrium in the presence of both fundamental (i.e. idiosyncratic and aggregate) and self-fulfilling uncertainty features less insurance against idiosyncratic uncertainty (i.e. lower c) and more insurance against aggregate uncertainty (i.e. higher $c_L(0)$) than what the depositors would obtain in a banking equilibrium with perfect information. As a consequence, a final question naturally regards whether the banking equilibrium with runs is "viable", in the sense of being able to provide a better allocation than the one that the depositors would obtain in the autarkic equilibrium, even in the presence of self-fulfilling uncertainty. In that respect, the same argument employed for the equilibrium with perfect information holds true in the banking equilibrium with runs: The bank can always choose to invest all deposits in liquidity, as the agents do in autarky. Put differently, in the banking equilibrium with runs the autarkic allocation is feasible, but is not chosen. Thus, it must be the case that the allocation of the banking equilibrium with runs Pareto-dominates the autarkic allocation.

5 Liquidity regulation

The argument of the previous section raises the question of what can actually make the banking system completely immune from financial fragility as in the equilibrium with perfect information, and whether that is a desirable option. Specifically, we focus on the effectiveness of a liquidity regulation that imposes "narrow banking". According to Pennacchi (2012), "a narrow bank is a financial institution that issues demandable liabilities and invests in assets that have little or no nominal interest rate and credit risk". A proposal along those lines by a group of University of Chicago economists in the 1930s (since then called the "Chicago Plan") has recently regained momentum (Benes and Kumhof, 2012; Cochrane, 2014). Its intended aim is to gain a better control of the credit cycle by reducing harmful liquidations, and eliminate bank runs by forcing the banks to hold an amount of cash reserves equal to their demand deposits (Fisher, 1936).

We study narrow banking in our framework by imposing on the banking problem the constraint $L \geq c$, i.e. such that the bank holds sufficient liquidity to pay early consumption to all depositors, even in the case of a run. Remember that, under both pecking orders, the total fraction of depositors that can be served before bankruptcy is $n^{**} = (L+r(1-L))/c$. Hence, imposing the narrow-banking constraint $L \geq c$ makes n^{**} larger than or equal to 1 under both pecking orders. In other words, narrow banking rules out self-fulfilling runs, as all depositors anticipate that the bank holds sufficient liquidity to pay early consumption to all of them. However, the effect of narrow banking is wider. In fact, remember the thresholds $\underline{\sigma}_1$ and $\underline{\sigma}_2$ for the lower dominance region under the two pecking orders in (20) and (26), respectively. If $L \geq c$, it is easy to see that both thresholds become smaller than or equal to 0. Thus, imposing the narrow-banking constraint $L \geq c$ makes the bank immune to self-fulfilling as well as fundamental runs.

To sum up, narrow banking is a business model that has the advantage of being completely run-proof. To characterize its equilibrium deposit contract and asset portfolio, we solve the following problem:

$$\max_{L,c} \int_0^1 \left[\lambda u(c) + (1-\lambda)\left[pu\left(\frac{R(1-L)+L-\lambda c}{1-\lambda}\right) + (1-p)u\left(\frac{L-\lambda c}{1-\lambda}\right)\right]\right] dp, \tag{39}$$

subject to the narrow-banking constraint $L \geq c$, and to $L \leq 1$. We use the first-order conditions of the problem to characterize the following Lemma:

Lemma 7. *The narrow-banking equilibrium satisfies:*

$$u'(c) = \mathbb{E}[p]Ru'(c_L(R)) + (1-\lambda)\xi + \mu, \tag{40}$$

where ξ and μ are the Lagrange multipliers on $L \geq c$ and $L \leq 1$, respectively.

Proof. In the text above. ■

Intuitively, the Lemma states that under narrow banking both sides of bank's balance sheets are distorted with respect to an equilibrium with perfect information, because on the one hand the constraint $L \geq c$ makes the bank immune from self-fulfilling uncertainty, but on the other it forces the bank to hold more liquidity than the one that it would need against fundamental uncertainty only.

From here, there are two possible cases, depending on whether the constraint $L \leq 1$ is binding or not. Assume first that $L^{NB} < 1$. Then, it must be the case that the narrow-banking constraint is binding. In fact, if it were slack, the narrow-banking equilibrium would be equivalent to the equilibrium with perfect information. However, that would mean $L^{NB} > c^{NB} = c^{PI} > 1$, which is a contradiction. Hence, if $L^{NB} < 1$ then $L^{NB} = c^{NB}$. This would yield the equilibrium allocation $c^{NB} = c_L^{NB}(0) = L^{NB} < 1$ and:

$$c_L^{NB}(R) = \frac{R(1 - L^{NB})}{1 - \lambda} + L^{NB} > 1, \tag{41}$$

with L^{NB} characterized by the equilibrium condition:

$$[\lambda + (1 - \lambda)\mathbb{E}[p]]u'(L^{NB}) = \mathbb{E}[p](R - 1 + \lambda)u'\left(c_L^{NB}(R)\right). \tag{42}$$

By the implicit function theorem, the previous expression shows that L^{NB} is increasing in λ: The higher the probability of the idiosyncratic shock is, the higher the amount of liquidity that a narrow bank would hold.

If instead the constraint $L \leq 1$ is binding and $L^{NB} = 1$, the first-order condition of the narrow-banking problem with respect to c yields the Lagrange multiplier:

$$\xi = u'(c) - u'\left(\frac{1 - \lambda c}{1 - \lambda}\right). \tag{43}$$

If $c < 1$, the multiplier is strictly positive, by the concavity of $u(c)$. Yet, that would be consistent with the equilibrium only if $c = 1$, which is a contradiction. In a similar way, c cannot be larger than 1, because that would violate the narrow-banking constraint. Hence, it must be the case that $c^{NB} = 1$. Together with $L^{NB} = 1$, this yields the equilibrium allocation $c^{NB} = L^{NB} = 1 = c_L^{NB}(0) = c_L^{NB}(R)$, which is equivalent to the autarkic equilibrium. In other words, under narrow banking the autarkic allocation is not only feasible, as in the banking problems of the previous sections, but also a possible equilibrium.

As a consequence of the previous result, analyzing the viability of narrow banking becomes a meaningful exercise. Put differently, would a narrow bank choose an equilibrium equivalent to autarky or not? Clearly, if that was the case, $L^{NB} = 1$ and the depositors' expected welfare would be equal to $u(1)$. If instead $L^{NB} < 1$, the depositors' expected welfare would be:

$$W^{NB} = \lambda u(c^{NB}) + (1-\lambda)\int_0^1 \Big[pu(c_L^{NB}(R)) + (1-p)u(c_L^{NB}(0))\Big]dp, \tag{44}$$

with:

$$\frac{\partial W^{NB}}{\partial \lambda} = u(c^{NB}) - \Big[\mathbb{E}[p]u(c_L^{NB}(R)) + (1-\mathbb{E}[p])u(c_L^{NB}(0))\Big]. \tag{45}$$

This derivative is negative as $\mathbb{E}[p] < 1$, $c^{NB} = c_L^{NB}(0) = L^{NB}$ and $c_L^{NB}(R) > L^{NB}$. To sum up, this result means that there must exist a probability of the idiosyncratic shock $\bar{\lambda}$ below which $L^{NB} < 1$ and $W^{NB} > u(1)$, i.e. narrow banking is viable. Above $\bar{\lambda}$, we must instead have that $L^{NB} = 1$ (as we proved that L^{NB} is increasing in λ) and $W^{NB} = u(1)$, so narrow banking is not viable.

Proposition 3. *If λ is sufficiently large, narrow banking is not viable.*

Proof. In the text above. ■

Intuitively, if the probability of the idiosyncratic shock is sufficiently large, a narrow bank is forced to be fully liquid, and that makes the narrow banking equilibrium equivalent to autarky. Therefore, a narrow bank, despite being immune from financial fragility, is at most redundant, because the agents would be as well off without it as with it.

6 Concluding remarks

With the present paper, we propose a novel mechanism through which financial fragility, in the form of self-fulfilling runs, forces the banks to hold extra precautionary liquidity. To this end, we study an environment in which banks manage both sides of their balance sheets to provide insurance against fundamental (i.e. idiosyncratic and aggregate) and self-fulfilling uncertainty at the same time. Our results characterize the conditions for the emergence of precautionary liquidity and extra precautionary liquidity, defined in comparison to suitable benchmarks. Moreover, they characterize the optimal pecking order that the banks follow when serving excessive depositors' withdrawals. Interestingly, this latter result could be extended to other financial intermediaries subject to strategic withdrawals. In that sense, it could represent a reconciliation (based on asset illiquidity and depositors' risk aversion) of the contrasting empirical evidence on mutual funds' cushioning versus hoarding, that in principle could be even brought to test in the data.

Finally, the clear characterization of the banks' liquidity management problem allows us to show the conditions under which liquidity regulation might harm the viability of the banking system. This is an example of a more general feature of our analysis: In the present framework, there is no failure of the fundamental theorems of welfare economics that justifies the introduction of liquidity regulation. More precisely, a constrained social planner subject to the same informational frictions of the banks (and therefore to self-fulfilling uncertainty) would not be able to offer a welfare-improving allocation over the one offered by the banks themselves. Thus, liquidity regulation in this framework would always be detrimental for welfare from a second-best perspective, unless we introduce some form of friction or market failure, for example a fire-sale externality that endogenizes the illiquidity of the productive asset in the spirit of Shleifer and Vishny (1992).

We see two more natural extensions to our work. First, the channel connecting banks' liquidity management and financial fragility might cause real effects on the long-run accumulation of capital and on economic growth, that are worthwhile analyzing in a dynamic model. Second, we could extend the present framework to analyze the interaction between ex-ante liquidity requirements and ex-post liquidity injections, and its effect on financial fragility and banks' liquidity management. In principle, we expect such policy measures to be considerably effective at reducing the probability of self-fulfilling runs. However, the effect on banks' liquidity management might be non-trivial, as the liquidity injections might strengthen or weaken the effect of liquidity on the probability of a run. We keep all these issues open to future research.

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Appendices

A Proofs

Proof of Lemma 2. Attach the Lagrange multipliers μ to the liquidity constraint (7) and ξ to the non-negativity constraint of D. The first-order conditions of the program are:

$$c: \qquad u'(c) - \int_0^1 \Big[pu'(c_L(R)) + (1-p)u'(c_L(0))\Big]dp - \mu = 0, \tag{46}$$

$$L: \qquad \int_0^1 \Big[pu'(c_L(R))(1-R) + (1-p)u'(c_L(0))\Big]dp + \mu = 0, \tag{47}$$

$$D: \qquad \int_0^1 \Big[pu'(c_L(R))(r-R) + (1-p)u'(c_L(0))r\Big]dp + r\mu + \xi = 0, \tag{48}$$

where $c_L(R)$ and $c_L(0)$ are the state-dependent amounts of late consumption. For the first part of the lemma, plug (47) into (48) and solve for the Lagrange multiplier:

$$\xi = (1-r)\int_0^1 \Big[pu'(c_L(R))R + (1-p)u'(c_L(0))\Big]. \tag{49}$$

This is clearly positive, implying that $D = 0$ by complementary slackness. For the second part of the lemma, assume that $c_L(0) \approx 0^+$. This implies that $L - \lambda c \approx 0^+$, and $\mu = 0$ by complementary slackness. For the first-order condition with respect to L to hold given that $u'(c_L(0)) \to +\infty$ by the Inada conditions, it has to be the case that $u'(c_L(R)) \to +\infty$ as well, meaning that also $c_L(R) \approx 0^+$. As a consequence, for the first-order condition with respect to c to hold, also $u'(c) \to +\infty$. Hence, $c \approx 0^+$, implying that $L \approx 0^+$. However, $c_L(R) \approx 0^+$ implies that $L + D \approx 1$, which leads to a contradiction. Finally, use (46) and (47) to derive (11).

As far as the third part of the Lemma is concerned, rewrite the bank budget constraints as:

$$L + Y = 1, \tag{50}$$

$$L = \lambda c + Q, \tag{51}$$

$$c_L(R)(1-\lambda) = RY + Q, \tag{52}$$

$$c_L(0)(1-\lambda) = Q, \tag{53}$$

where Y is the amount invested at date 0 in the productive asset, and Q is precautionary liquidity. Aggregate the budget constraints to derive the intertemporal budget constraint:

$$\lambda c + (1-\lambda)\left[\frac{c_L(R)}{R} + \left(1 - \frac{1}{R}\right)c_L(0)\right] = 1. \tag{54}$$

Assume that $c^{PI} \leq 1$. By the Euler equation (11):

$$\mathbb{E}[p]R = \frac{u'(c^{PI})}{u'(c_L^{PI}(R))} > \frac{c_L^{PI}(R)}{c^{PI}} \geq c_L^{PI}(R) > c_L^{PI}(0), \tag{55}$$

where the first inequality is a consequence of the assumption of relative risk aversion being larger than one,[13] the second inequality comes from $c^{PI} \leq 1$, and the third inequality holds by construction. Thus, the term in the square brackets of the intertemporal budget constraint (54), as a linear combination of two terms smaller than $\mathbb{E}[p]R$, must be also smaller than $\mathbb{E}[p]R$. Hence:

$$\lambda c + (1-\lambda)\mathbb{E}[p]R > 1. \tag{56}$$

By continuity, there exists a λ sufficiently large such that this inequality does not hold. Hence, under this condition, we get a contradiction, implying that $c^{PI} > 1$. By the Euler equation, this also implies that $c_L^{PI}(R) > 1$. Finally, for this to be consistent with the intertemporal budget constraint (54), it must be the case that $c_L^{PI}(0) < 1$. This ends the proof. ■

Proof of Lemma 3. Rewrite the derivative as:

$$\frac{\partial v_1}{\partial n} = \sigma u'(c_L(R,n)) \frac{R\left(1 - L - \frac{nc}{r}\right) + L - \frac{R}{r}c(1-n)}{(1-n)^2} + (1-\sigma)u'(c_L(0,n))\frac{L}{(1-n)^2}. \tag{57}$$

This expression is negative whenever:

$$\sigma u'(c_L(R,n))R\left(\frac{c}{r} - 1\right) > L\Big[\sigma u'(c_L(R,n))(1-R) + (1-\sigma)u'(c_L(0,n))\Big] >$$
$$> Lu'(c_L(R,n))(1-R), \tag{58}$$

where the last inequality follows from the term in the square bracket being a linear combination of two terms, with $u'(c_L(0,n)) > u'(c_L(R,n))(1-R)$. Hence, the derivative is negative, provided that:

$$\sigma R\left(\frac{c}{r} - 1\right) > L(1-R). \tag{59}$$

As $R > 1$ by assumption, this last expression is always true if $c > r$ (as it turns out in the banking equilibrium). ■

Proof of Lemma 4. The threshold signal σ_1^* is the value of σ that makes a late consumer indif-

[13] To see that, rewrite $-u''(c)c/u'(c) > 1$ as $-u''(c)/u'(c) > 1/c$. This, in turn, means that $-(log[u'(c)])' > (log[c])'$. Integrate between c_1 and $c_2 > c_1$ so as to obtain $log[u'(c_1)] - log[u'(c_2)] > log[c_2] - log[c_1]$. Once taken the exponent, the last expression gives $u'(c_1)/u'(c_2) > c_2/c_1$. If $c_1 > c_2$, the inequality is reversed.

ferent between waiting or running, given his or her posterior beliefs:

$$\int_{\lambda}^{n_1^*} \Big[\sigma_1^* u(c_L(R,n)) + (1-\sigma_1^*)u(c_L(0,n))\Big] dn + \int_{n_1^*}^{n_1^{**}} u(c_L^L(n))dn = \int_{\lambda}^{n_1^{**}} u(c)dn + \int_{n_1^{**}}^{1} u\left(c^B(n)\right) dn. \tag{60}$$

Rearranging this expression, we get the threshold signal in (24). The derivative of the threshold signal σ_1^* with respect to c reads:

$$\frac{\partial \sigma_1^*}{\partial c} = \frac{1}{\int_{\lambda}^{n_1^*} \Big[u(c_L(R,n)) - u(c_L(0,n))\Big] dn} \times \\ \times \left[(n_1^{**} - \lambda)u'(c) + \int_{n_1^*}^{n_1^{**}} u'(c_L^L(n)) \frac{n}{1-n} dn + \sigma_1^* \int_{\lambda}^{n_1^*} u'(c_L(R,n)) \frac{Rn}{r(1-n)} dn \right], \tag{61}$$

which is always positive as the utility function is increasing. In a similar way, the derivative of the threshold signal σ_1^* with respect to L reads:

$$\frac{\partial \sigma_1^*}{\partial L} = \frac{1}{\int_{\lambda}^{n_1^*} \Big[u(c_L(R,n)) - u(c_L(0,n))\Big] dn} \times \\ \times \left[\int_{n_1^{**}}^{1} u'(c^B(n)) \frac{1-r}{n} dn - \int_{\lambda}^{n_1^*} u'(c_L(0,n)) \frac{1}{1-n} dn - \int_{n_1^*}^{n_1^{**}} u'(c_L^L(n)) \frac{1-r}{1-n} dn + \\ - \sigma_1^* \int_{\lambda}^{n_1^*} \left[u'(c_L(R,n)) \frac{1-R}{1-n} - u'(c_L(0,n)) \frac{1}{1-n}\right] dn \right]. \tag{62}$$

Notice that $\lim_{n \to n_1^{**}} u'(c_L^L(n)) = \lim_{c \to 0} u'(c) = \psi^{-\gamma}$ which is large but finite. Hence, $\partial \sigma_1^* / \partial L$ is negative. This ends the proof. ■

Proof of Lemma 5. By definition, $u(c)$ is strictly concave on an open internal X if and only if:

$$u(x) - u(y) < u'(y)(x-y), \tag{63}$$

for any x and y in X. Hence, when $\lambda \leq n < n_2^*$, it must be the case that:

$$\frac{\partial v_2}{\partial n} = \sigma u'(c_L(R,n)) \frac{c_L(R,n) - c}{1-n} - (1-\sigma)u'(c_L(0,n)) \frac{c - c_L(0,n)}{1-n} <$$

$$
\begin{aligned}
&< \sigma \frac{u(c_L(R,n)) - u(c)}{1-n} - (1-\sigma)\frac{u(c) - u(c_L(0,n))}{1-n} = \\
&= \frac{\sigma u(c_L(R,n)) + (1-\sigma)u(c_L(0,n)) - u(c)}{1-n} = \frac{v_2(n)}{1-n}
\end{aligned} \tag{64}
$$

Thus, whenever $v_2(n) \leq 0$, the derivative is negative. This ends the proof. ■

Proof of Lemma 6. The threshold signal σ_2^* is the value of σ that equalizes:

$$
\begin{aligned}
&\int_{\lambda}^{n_2^*} \left[\sigma_2^* u\left(\frac{R(1-L)+L-nc}{1-n}\right) + (1-\sigma_2^*)u\left(\frac{L-nc}{1-n}\right)\right]dn + \\
&+ \int_{n_2^*}^{n^{**}} \sigma_2^* u\left(\frac{R\left(1-L-\frac{nc-L}{r}\right)}{1-n}\right) dn = \int_{\lambda}^{n^{**}} u(c)dn + \int_{n^{**}}^{1} u\left(\frac{L+r(1-L)}{n}\right) dn.
\end{aligned} \tag{65}
$$

Rearranging this expression, we get the threshold signal σ_2^* in (33). The derivative of the threshold signal σ_2^* with respect to c reads:

$$
\begin{aligned}
\frac{\partial \sigma_2^*}{\partial c} &= \frac{1}{\int_{\lambda}^{n_2^*}\left[u\left(\frac{R(1-L)+L-nc}{1-n}\right) - u\left(\frac{L-nc}{1-n}\right)\right]dn + \int_{n_2^*}^{n^{**}} u\left(\frac{R\left(1-L-\frac{nc-L}{r}\right)}{1-n}\right)dn} \times \\
&\times \left[(n^{**}-\lambda)u'(c) + \sigma_2^*\left[\int_{\lambda}^{n_2^*} u'(c_L(R,n))\frac{n}{1-n}dn + \int_{n_2^*}^{n^{**}} u'(c_L^D(R,n))\frac{Rn}{r(1-n)}dn\right] + \right. \\
&\left. + (1-\sigma_2^*)\int_{\lambda}^{n_2^*} u'(c_L(0,n))\frac{n}{1-n}dn\right].
\end{aligned} \tag{66}
$$

This derivative is positive, because the utility function is increasing and $\sigma_2^* \leq 1$. The derivative of the threshold signal σ_2^* with respect to L instead reads:

$$
\begin{aligned}
\frac{\partial \sigma_2^*}{\partial L} &= \frac{1}{\int_{\lambda}^{n_2^*}\left[u\left(\frac{R(1-L)+L-nc}{1-n}\right) - u\left(\frac{L-nc}{1-n}\right)\right]dn + \int_{n_2^*}^{n^{**}} u\left(\frac{R\left(1-L-\frac{nc-L}{r}\right)}{1-n}\right)dn} \times \\
&\times \left[\int_{n^{**}}^{1} u'(c^B(n))\frac{1-r}{n}dn - \int_{\lambda}^{n_2^*} u'(c_L(0,n))\frac{1}{1-n}dn + \right. \\
&- \sigma_2^*\left[\int_{\lambda}^{n_2^*} u'(c_L(R,n))\frac{1-R}{1-n}dn - \int_{\lambda}^{n_2^*} u'(c_L(0,n))\frac{1}{1-n}dn + \right. \\
&\left.\left. + \int_{n_2^*}^{n^{**}} u'(c_L^D(R,n))R\left(\frac{1}{r}-1\right)\frac{1}{1-n}dn\right]\right].
\end{aligned} \tag{67}
$$

As $\sigma_2^* < 1$ and:

$$\lim_{n \to n_2^*} u'(c_L(0,n)) = \lim_{n \to n^{**}} u'(c_L^D(R,n)) = \lim_{c \to 0} u'(c) = \psi^{-gamma} \tag{68}$$

under CRRA, (67) is negative. This ends the proof. ■

Proof of Proposition 1. We study σ_1^* and σ_2^* as functions of the recovery rate r. As $r \to \underline{r} = \lambda c/(1-L)$, we have that $n_1^* \to \lambda$ and the first interval of $v_1(n)$ reduces to zero. Thus, the expected value of waiting versus running under the pecking order {Liquidation, Liquidity} becomes:

$$\mathbb{E}[v_1(n)] = \int_{\lambda}^{n^{**}} \left[u\left(\frac{L + r(1-L) - nc}{1-n}\right) - u(c) \right] dn - \int_{n^{**}}^{1} u\left(\frac{L + r(1-L)}{n}\right) dn. \tag{69}$$

This expression is always negative, as the numerator of σ_1^* must be positive. Hence, σ_1^* is constant and equal to 1 in the interval $[0, \underline{r}]$. In the interval $[\underline{r}, 1]$, instead, the threshold signal σ_1^* is a decreasing and convex function of the recovery rate r. To see that, calculate:

$$\begin{aligned}
\frac{\partial \sigma_1^*}{\partial r} = & \frac{1}{\left[\int_{\lambda}^{n_1^*} \Big[u(c_L(R,n)) - u(c_L(0,n)) \Big] dn \right]^2} \times \\
& \times \left[\left[\int_{n^{**}}^{1} u'(c^B(n)) \frac{1-L}{n} dn - \int_{n_1^*}^{n^{**}} u'(c_L^L(n)) \frac{1-L}{1-n} dn \right] \times \right. \\
& \times \left[\int_{\lambda}^{n_1^*} \Big[u(c_L(R,n)) - u(c_L(0,n)) \Big] dn \right] - \left[- \int_{\lambda}^{n_1^*} u'(c_L(R,n)) \frac{Rnc}{r^2(1-n)} dn \right] \times \\
& \times \left. \left[\int_{\lambda}^{n_1^{**}} u(c) dn + \int_{n_1^{**}}^{1} u\left(c^B(n)\right) dn - \int_{\lambda}^{n_1^*} u(c_L(0,n)) dn - \int_{n_1^*}^{n_1^{**}} u(c_L^L(n)) dn \right] \right].
\end{aligned} \tag{70}$$

By the Inada conditions, we know that $\lim_{n \to n^{**}} u'(c_L^L(n)) = \lim_{c \to 0} u'(c) = +\infty$. Hence, the derivative must be negative. Crucial for this result is the fact that, for any pecking order j, $v_j(n)$ is continuous everywhere, but has kinks at n_j^* and n^{**}, so it is not differentiable at those points.

To show that the threshold signal σ_1^* is instead a convex function of r, calculate:

$$\frac{\partial^2 \sigma_1^*}{\partial r^2} = \frac{1}{\left[\int_{\lambda}^{n_1^*} \Big[u(c_L(R,n)) - u(c_L(0,n)) \Big] dn \right]^4} \times$$

$$
\begin{aligned}
&\times \Bigg[\Bigg[\Bigg[-\frac{1-L}{c}u'(c)\frac{1-L}{n^{**}} + \int_{n^{**}}^{1} u''(c^B(n))\left(\frac{1-L}{c}\right)^2 dn + \\
&- \int_{n_1^*}^{n^{**}} u''(c_L^L(n))\left(\frac{1-L}{1-n}\right)^2 dn + \frac{1-L}{c}u'\left(\frac{L}{1-n}\right)\frac{1-L}{1-n_1^*}\Bigg] \times \\
&\times \left[\int_{\lambda}^{n_1^*}\Big[u(c_L(R,n)) - u(c_L(0,n))\Big]dn\right] + \left[\int_{\lambda}^{n_1^*} u'(c_L(R,n))\frac{Rnc}{r^2(1-n)}dn\right] \times \\
&\times \left[\int_{n^{**}}^{1} u'(c^B(n))\frac{1-L}{n}dn - \int_{n_1^*}^{n^{**}} u'(c_L^L(n))\frac{1-L}{1-n}dn\right] + \\
&- \Bigg[\frac{1-L}{c}u'\left(\frac{L}{1-n_1^*}\right)\frac{R(1-L)}{r(1-n_1^*)}dn + \\
&+ \int_{\lambda}^{n_1^*}\left[u''(c_L(R,n))\left(\frac{Rnc}{r^2(1-n)}\right)^2 - 2u'(c_L(R,n))\frac{Rnc}{r^3(1-n)}\right]dn\Bigg] \times \\
&\times \left[\int_{\lambda}^{n_1^{**}} u(c)dn + \int_{n_1^{**}}^{1} u\left(c^B(n)\right)dn - \int_{\lambda}^{n_1^*} u(c_L(0,n))dn - \int_{n_1^*}^{n_1^{**}} u(c_L^L(n))dn\right] + \\
&- \left[\int_{\lambda}^{n_1^*} u'(c_L(R,n))\frac{Rnc}{r^2(1-n)}dn\right] \times \\
&\times \left[\int_{n^{**}}^{1} u'(c^B(n))\frac{1-L}{n}dn - \int_{n_1^*}^{n^{**}} u'(c_L^L(n))\frac{1-L}{1-n}dn\right]\Bigg] \times \\
&\times \left[\int_{\lambda}^{n_1^*}\Big[u(c_L(R,n)) - u(c_L(0,n))\Big]dn\right]^2 + \\
&- \Bigg[\left[\int_{n^{**}}^{1} u'(c^B(n))\frac{1-L}{n}dn - \int_{n_1^*}^{n^{**}} u'(c_L^L(n))\frac{1-L}{1-n}dn\right] \times \\
&\times \left[\int_{\lambda}^{n_1^*}\Big[u(c_L(R,n)) - u(c_L(0,n))\Big]dn\right] + \\
&- \left[\int_{\lambda}^{n_1^{**}} u(c)dn + \int_{n_1^{**}}^{1} u\left(c^B(n)\right)dn - \int_{\lambda}^{n_1^*} u(c_L(0,n))dn - \int_{n_1^*}^{n_1^{**}} u(c_L^L(n))dn\right] \times \\
&\times \left[\int_{\lambda}^{n_1^*} u'(c_L(R,n))\frac{Rnc}{r^2(1-n)}dn\right]\Bigg] \times \\
&\times 2\left[\int_{\lambda}^{n_1^*}\Big[u(c_L(R,n)) - u(c_L(0,n))\Big]dn\right]\int_{\lambda}^{n_1^*} u'(c_L(R,n))\frac{Rnc}{r^2(1-n)}dn\Bigg].
\end{aligned} \tag{71}
$$

By definition of CRRA utility, and for ψ arbitrarily close to but larger than 0:

$$\frac{u''(x)}{u'(x)} = -\frac{\gamma}{x}, \tag{72}$$

where γ is the constant coefficient of relative risk aversion. This implies that:

$$\lim_{n \to n^{**}} \frac{u''(c_L^L(n))}{u'(c_L^L(n))} = \lim_{x \to 0} \frac{u''(x)}{u'(x)} = -\lim_{x \to 0} \frac{\gamma}{x} = -\infty. \tag{73}$$

Hence, $u''(c)$ goes to $-\infty$ at a speed faster than the one at which $u'(c)$ goes to $+\infty$, when $c \to 0$. This, together with the Inada conditions, ensures that the second derivative is positive, meaning that σ_1^* is a convex function of r.

In contrast, σ_2^* at $r = 0$ is always lower than 1 if R is sufficiently large. To see that, notice that:

$$\sigma_2^*|_{r=0} = \frac{\int_\lambda^{\frac{L}{c}} u(c)dn + \int_{\frac{L}{c}}^1 u\left(\frac{L}{n}\right) dn - \int_\lambda^{\frac{L}{c}} u\left(\frac{L-nc}{1-n}\right) dn}{\int_\lambda^{\frac{L}{c}} \left[u\left(\frac{R(1-L)+L-nc}{1-n}\right) - u\left(\frac{L-nc}{1-n}\right)\right] dn}. \tag{74}$$

This expression is lower than 1 if:

$$\int_\lambda^{\frac{L}{c}} u(c)dn + \int_{\frac{L}{c}}^1 u\left(\frac{L}{n}\right) dn < \int_\lambda^{\frac{L}{c}} u\left(\frac{R(1-L)+L-nc}{1-n}\right) dn. \tag{75}$$

This condition is true if R is sufficiently high, given that $R > c$ must hold. In fact, under CRRA utility and with ψ arbitrarily close to but larger than 0, (75) reads:

$$\int_\lambda^{\frac{L}{c}} \left[\frac{\left(\frac{R(1-L)+L-nc}{1-n}\right)^{1-\gamma}}{\gamma - 1} - \frac{c^{1-\gamma}}{\gamma-1}\right] dn - \int_{\frac{L}{c}}^1 \frac{\left(\frac{L}{n}\right)^{1-\gamma}}{\gamma-1} dn < 0. \tag{76}$$

This is equivalent to:

$$\int_\lambda^{\frac{L}{c}} \left[\left(\frac{(1-L) + \frac{L-nc}{R}}{1-n}\right)^{1-\gamma} R^{1-\gamma} - c^{1-\gamma}\right] dn - \int_{\frac{L}{c}}^1 \left(\frac{L}{n}\right)^{1-\gamma} dn < 0. \tag{77}$$

Multiply the previous expression by $R^{\gamma-1}$, and rewrite it as:

$$\int_\lambda^{\frac{L}{c}} \left[\left(\frac{(1-L) + \frac{L-nc}{R}}{1-n}\right)^{1-\gamma} - \left(\frac{c}{R}\right)^{1-\gamma}\right] dn - \int_{\frac{L}{c}}^1 \left(\frac{L}{n}\right)^{1-\gamma} R^{\gamma-1} dn < 0. \tag{78}$$

As $R > c$, $\frac{c}{R}$ is bounded. Therefore, this condition is always satisfied for $R \to \infty$, as the last integral goes to $-\infty$. By continuity, there must be a sufficiently large and finite value of R such that this is also true.

Having proved that the threshold signal $\sigma_2^* < 1$ at $r = 0$, we want to show that it is also a decreasing and convex function of the recovery rate r. To this end, we first calculate:

$$\begin{aligned}
\frac{\partial \sigma_2^*}{\partial r} = & \left[\int_{\lambda}^{n_2^*}\left[u\left(\frac{R(1-L)+L-nc}{1-n}\right)-u\left(\frac{L-nc}{1-n}\right)\right]dn+\int_{n_2^*}^{n^{**}}u\left(\frac{R\left(1-L-\frac{nc-L}{r}\right)}{1-n}\right)dn\right]^{-2}\times \\
& \times\left[\left[\int_{n^{**}}^{1}u'(c^B(n))\frac{1-L}{n}dn\right]\times\right. \\
& \times\left[\int_{\lambda}^{n_2^*}\left[u\left(\frac{R(1-L)+L-nc}{1-n}\right)-u\left(\frac{L-nc}{1-n}\right)\right]dn+\int_{n_2^*}^{n^{**}}u\left(\frac{R\left(1-L-\frac{nc-L}{r}\right)}{1-n}\right)dn\right]+ \\
& -\left[\int_{n_2^*}^{n^{**}}u'\left(\frac{R\left(1-L-\frac{nc-L}{r}\right)}{1-n}\right)\frac{Rnc}{r^2(1-n)}dn\right]\times \\
& \left.\times\left[\int_{\lambda}^{n^{**}}u(c)dn+\int_{n^{**}}^{1}u\left(c^B(n)\right)dn-\int_{\lambda}^{n_2^*}u\left(\frac{L-nc}{1-n}\right)dn\right]\right].
\end{aligned} \tag{79}$$

By the same considerations as before regarding the Inada conditions, notice that:

$$\lim_{n\to n^{**}}u'\left(\frac{R\left(1-L-\frac{nc-L}{r}\right)}{1-n}\right)=+\infty. \tag{80}$$

Hence, the derivative must be negative.

To show that the threshold signal σ_2^* is instead a convex function of r, calculate:

$$\begin{aligned}
\frac{\partial^2 \sigma_2^*}{\partial r^2} = & \left[\int_{\lambda}^{n_2^*}\left[u\left(\frac{R(1-L)+L-nc}{1-n}\right)-u\left(\frac{L-nc}{1-n}\right)\right]dn+\int_{n_2^*}^{n^{**}}u\left(\frac{R\left(1-L-\frac{nc-L}{r}\right)}{1-n}\right)dn\right]^{-4}\times \\
& \times\left[\left[\left[-\frac{1-L}{c}u'(c)\frac{1-L}{n^{**}}+\int_{n^{**}}^{1}u''(c^B(n))\left(\frac{1-L}{n}\right)^2dn\right]\times\right.\right. \\
& \times\left[\int_{\lambda}^{n_2^*}\left[u\left(\frac{R(1-L)+L-nc}{1-n}\right)-u\left(\frac{L-nc}{1-n}\right)\right]dn+\int_{n_2^*}^{n^{**}}u\left(\frac{R\left(1-L-\frac{nc-L}{r}\right)}{1-n}\right)dn\right]+ \\
& +\left[\int_{n^{**}}^{1}u'(c^B(n))\frac{1-L}{n}dn\right]\left[\int_{n_2^*}^{n^{**}}u'\left(\frac{R\left(1-L-\frac{nc-L}{r}\right)}{1-n}\right)\frac{Rnc}{r^2(1-n)}dn\right]+ \\
& -\left[\int_{\lambda}^{n^{**}}u(c)dn+\int_{n^{**}}^{1}u\left(c^B(n)\right)dn-\int_{\lambda}^{n_2^*}u\left(\frac{L-nc}{1-n}\right)dn\right]\times \\
& \times\left[\int_{n_2^*}^{n^{**}}\left[u''\left(\frac{R\left(1-L-\frac{nc-L}{r}\right)}{1-n}\right)\left(\frac{Rnc}{r^2(1-n)}\right)^2-2u'\left(\frac{R\left(1-L-\frac{nc-L}{r}\right)}{1-n}\right)\frac{Rnc}{r^3(1-n)}\right]dn\right]+
\end{aligned}$$

$$
\begin{aligned}
&-\left[\int_{n_2^*}^{n^{**}} u'\left(\frac{R\left(1-L-\frac{nc-L}{r}\right)}{1-n}\right)\frac{Rnc}{r^2(1-n)}dn\right]\times \\
&\times\left[\int_{n^{**}}^{1} u'(c^B(n))\frac{1-L}{n}dn\right]\Bigg]\times \\
&\times\left[\int_{\lambda}^{n_2^*}\left[u\left(\frac{R(1-L)+L-nc}{1-n}\right)-u\left(\frac{L-nc}{1-n}\right)\right]dn+\int_{n_2^*}^{n^{**}} u\left(\frac{R\left(1-L-\frac{nc-L}{r}\right)}{1-n}\right)dn\right]^2+ \\
&-2\left[\int_{\lambda}^{n_2^*}\left[u\left(\frac{R(1-L)+L-nc}{1-n}\right)-u\left(\frac{L-nc}{1-n}\right)\right]dn+\int_{n_2^*}^{n^{**}} u\left(\frac{R\left(1-L-\frac{nc-L}{r}\right)}{1-n}\right)dn\right]\times \\
&\times\left[\int_{n_2^*}^{n^{**}} u'\left(\frac{R\left(1-L-\frac{nc-L}{r}\right)}{1-n}\right)\frac{Rnc}{r^2(1-n)}dn\right]\times \\
&\times\Bigg[\left[\int_{n^{**}}^{1} u'(c^B(n))\frac{1-L}{n}dn\right]\times \\
&\times\left[\int_{\lambda}^{n_2^*}\left[u\left(\frac{R(1-L)+L-nc}{1-n}\right)-u\left(\frac{L-nc}{1-n}\right)\right]dn+\int_{n_2^*}^{n^{**}} u\left(\frac{R\left(1-L-\frac{nc-L}{r}\right)}{1-n}\right)dn\right]+ \\
&-\left[\int_{n_2^*}^{n^{**}} u'\left(\frac{R\left(1-L-\frac{nc-L}{r}\right)}{1-n}\right)\frac{Rnc}{r^2(1-n)}dn\right]\times \\
&\times\left[\int_{\lambda}^{n^{**}} u(c)dn+\int_{n^{**}}^{1} u\left(c^B(n)\right)dn-\int_{\lambda}^{n_2^*} u\left(\frac{L-nc}{1-n}\right)dn\right]\Bigg]\Bigg]. \qquad (81)
\end{aligned}
$$

By the same consideration regarding the Inada conditions, we get that this derivative is positive, meaning that σ_2^* is a convex function of r.

Being the two threshold signals σ_1^* and σ_2^* both decreasing and convex functions of the recovery rate r, to prove that they cross only once in the interval $[0,1]$ it suffices to prove that $\sigma_2^* > \sigma_1^*$ at $r=1$:

$$
\sigma_1^*|_{r=1} = \frac{\int_{\lambda}^{\frac{1}{c}} u(c)dn + \int_{\frac{1}{c}}^{1} u\left(\frac{1}{n}\right)dn - \int_{\lambda}^{\frac{1-L}{c}} u\left(\frac{L}{1-n}\right)dn - \int_{\frac{1-L}{c}}^{\frac{1}{c}} u\left(\frac{1-nc}{1-n}\right)dn}{\int_{\lambda}^{\frac{1-L}{c}}\left[u\left(\frac{R(1-L-nc)+L}{1-n}\right)-u\left(\frac{L}{1-n}\right)\right]dn}, \qquad (82)
$$

$$
\sigma_2^*|_{r=1} = \frac{\int_{\lambda}^{\frac{1}{c}} u(c)dn + \int_{\frac{1}{c}}^{1} u\left(\frac{1}{n}\right)dn - \int_{\lambda}^{\frac{L}{c}} u\left(\frac{L-nc}{1-n}\right)dn}{\int_{\lambda}^{\frac{L}{c}}\left[u\left(\frac{R(1-L)+L-nc}{1-n}\right)-u\left(\frac{L-nc}{1-n}\right)\right]dn + \int_{\frac{L}{c}}^{\frac{1}{c}} u\left(\frac{R(1-nc)}{1-n}\right)dn}. \qquad (83)
$$

Define as NUM_j and DEN_j the numerator and denominator of σ_j^*, respectively, for any pecking order $j = \{1, 2\}$. The following relationship holds:

$$NUM_1 = NUM_2 + \int_{\lambda}^{n_2^*} u\left(\frac{L - nc}{1 - n}\right) dn - \int_{\lambda}^{n_1^*} u\left(\frac{L}{1 - n}\right) dn - \int_{n_1^*}^{n^{**}} u\left(\frac{r(1 - L) + L - nc}{1 - n}\right) dn. \tag{84}$$

As a preliminary, step, we want to show that:

$$H \equiv \int_{\lambda}^{n_2^*} u\left(\frac{L - nc}{1 - n}\right) dn - \int_{\lambda}^{n_1^*} u\left(\frac{L}{1 - n}\right) dn - \int_{n_1^*}^{n^{**}} u\left(\frac{r(1 - L) + L - nc}{1 - n}\right) dn \tag{85}$$

is negative. If $n_1^* \leq n_2^*$, the previous expression can be re-written as:

$$\begin{aligned} H = & \int_{\lambda}^{n_1^*} u\left(\frac{L - nc}{1 - n}\right) dn + \int_{n_1^*}^{n_2^*} u\left(\frac{L - nc}{1 - n}\right) dn - \int_{\lambda}^{n_1^*} u\left(\frac{L}{1 - n}\right) dn + \\ & - \int_{n_1^*}^{n_2^*} u\left(\frac{r(1 - L) + L - nc}{1 - n}\right) dn - \int_{n_2^*}^{n^{**}} u\left(\frac{r(1 - L) + L - nc}{1 - n}\right) dn, \end{aligned} \tag{86}$$

which is clearly negative. In a similar way, if $n_1^* > n_2^*$, we can re-write:

$$\begin{aligned} H = & \int_{\lambda}^{n_2^*} u\left(\frac{L - nc}{1 - n}\right) dn - \int_{\lambda}^{n_2^*} u\left(\frac{L}{1 - n}\right) dn - \int_{n_2^*}^{n_1^*} u\left(\frac{L}{1 - n}\right) dn + \\ & - \int_{n_1^*}^{n^{**}} u\left(\frac{r(1 - L) + L - nc}{1 - n}\right) dn, \end{aligned} \tag{87}$$

which again is always negative. Thus, $NUM_1 < NUM_2$. Given this result, a sufficient condition for $\sigma_2^* \geq \sigma_1^*$ is $DEN_2 \leq DEN_1$, or:

$$\begin{aligned} f(c, L) = & \int_{\lambda}^{\frac{L}{c}} \left[u\left(\frac{R(1 - L) + L - nc}{1 - n}\right) - u\left(\frac{L - nc}{1 - n}\right)\right] dn + \\ & + \int_{\frac{L}{c}}^{\frac{1}{c}} u\left(\frac{R(1 - nc)}{1 - n}\right) dn - \int_{\lambda}^{\frac{1-L}{c}} \left[u\left(\frac{R(1 - L - nc) + L}{1 - n}\right) - u\left(\frac{L}{1 - n}\right)\right] dn \leq 0. \end{aligned} \tag{88}$$

We study how $f(c, L)$ changes with c and L. On the one side:

$$\begin{aligned} \frac{\partial f(c, L)}{\partial c} = & - \int_{\lambda}^{\frac{L}{c}} \left[u'\left(\frac{R(1 - L) + L - nc}{1 - n}\right) - u'\left(\frac{L - nc}{1 - n}\right)\right] \frac{n}{1 - n} dn + \\ & - \int_{\frac{L}{c}}^{\frac{1}{c}} u'\left(\frac{R(1 - nc)}{1 - n}\right) \frac{Rn}{1 - n} dn + \int_{\lambda}^{\frac{1-L}{c}} u'\left(\frac{R(1 - L - nc) + L}{1 - n}\right) \frac{Rn}{1 - n} dn. \end{aligned} \tag{89}$$

The sign of this derivative is positive. To see that, notice that $R(1-nc)/(1-n) > (L-nc)/(1-n)$. Hence, by the fact that the coefficient of relative risk aversion is larger than 1:[14]

$$\frac{u'\left(\frac{R(1-nc)}{1-n}\right)}{u'\left(\frac{L-nc}{1-n}\right)} < \frac{L-nc}{R(1-nc)}, \tag{90}$$

and this implies that:

$$u'\left(\frac{R(1-nc)}{1-n}\right)\frac{Rn}{1-n} < u'\left(\frac{L-nc}{1-n}\right)\frac{n}{1-n}\frac{L-nc}{1-nc} < u'\left(\frac{L-nc}{1-n}\right)\frac{n}{1-n}. \tag{91}$$

On the other side:

$$\begin{aligned}\frac{\partial f(c,L)}{\partial L} = &-\int_{\lambda}^{\frac{L}{c}}\left[u'\left(\frac{R(1-L)+L-nc}{1-n}\right)(R-1)+u'\left(\frac{L-nc}{1-n}\right)\right]\frac{1}{1-n}dn+\\ &+\int_{\lambda}^{\frac{1-L}{c}}\left[u'\left(\frac{R(1-L-nc)+L}{1-n}\right)(R-1)+u'\left(\frac{L}{1-n}\right)\right]\frac{1}{1-n}dn.\end{aligned} \tag{92}$$

This is negative because of the Inada Conditions, that make the second integral in the first line become large and negative. Since $f(c,L)$ is increasing in c and decreasing in L, a sufficient condition for it to be less than or equal to zero everywhere is that it is less than or equal to zero at $L^{\min} = \lambda$ and $c^{\max}$ when $L = \lambda$, which is $c^{\max} = 1$. At those points, the condition $f(c,L) \leq 0$ reads:

$$u(R)(1-\lambda) - \int_{\lambda}^{1-\lambda}\left[u\left(\frac{R(1-\lambda-n)+\lambda}{1-n}\right) - u\left(\frac{\lambda}{1-n}\right)\right]dn \leq 0. \tag{93}$$

Figure 9 numerically shows that condition (93) holds for high values of the coefficient of relative risk aversion.[15] This ends the proof. ■

Proof of Proposition 2. Attach the Lagrange multipliers μ to the liquidity constraint $L \geq \lambda c$. The first-order conditions of the program reads:

$$c: \quad -\frac{\partial\sigma^{BE}}{\partial c}\Delta U(c,L) + \lambda\int_{\sigma^{BE}}^{1}\left[u'(c) - \left[pu'(c_L(R)) + (1-p)u'(c_L(0))\right]\right]dp - \lambda\mu = 0, \tag{94}$$

[14] See footnote 13.

[15] We assume CRRA utility, with $\gamma > 1$, $\psi = 2$, $R = 2.01$ and $\lambda = .01$. The results are robust to different parameter choices. In particular, the choice of $\psi = 2$ is only for the sake of exposition: A value of ψ arbitrarily close to but larger than 0 would not qualitatively change the result in any way.

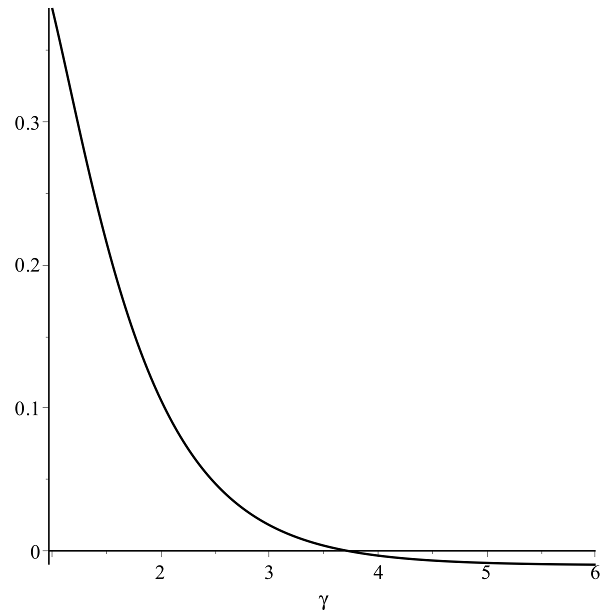

Figure 9: The condition (93) as a function of the coefficient of relative risk aversion.

$$
\begin{aligned}
L: \quad & -\frac{\partial \sigma^{BE}}{\partial L}\Delta U(c,L) + \sigma^{BE}(1-r)u'(L+r(1-L)) + \\
& + \int_{\sigma^{BE}}^{1} \Big[p u'(c_L(R))(1-R) + (1-p)u'(c_L(0)) \Big] dp + \mu = 0. \qquad (95)
\end{aligned}
$$

The same lines of reasoning employed in the proof of Lemma 2 apply here, so $c_L(0) \approx 0^+$ is not compatible with the first-order conditions, so in equilibrium $L > \lambda c$. Plugging (94) into (95) gives (37). Notice that in equilibrium it must be the case that $c < c_L(R)$, otherwise the thresholds of the lower dominance region under both pecking orders would be larger than or equal to 1. Moreover, For the sign of the strategic complementarity in the interval $[\lambda, n_2^*]$ we had to prove that $R(1-L)/(c-L) > 1$. This is satisfied by $c < c_L(R)$ by the concavity of $u(c)$. This also implies that $c < R$, thus confirming the condition for the existence of the upper dominance region. One final consequence of $R(1-L)/(c-L) > 1$ is that $c > c_L(0)$. To see that, assume not. However, $c \leq c_L(0)$ would imply $c \leq L$, which is a contradiction. This ends the proof. ■

Proof of Corollary 1. In order to characterize the sign of the distortion in (37) with respect to the banking equilibrium with perfect information, we start by deriving the sign of the sum of the marginal effects, for the pecking order {Liquidity; Liquidation}:[16]

$$\left[\frac{\partial \sigma_2^*}{\partial L}+\frac{1}{\lambda}\frac{\partial \sigma_2^*}{\partial c}\right]=\frac{1}{\int_\lambda^{n_2^*}\left[u\left(\frac{R(1-L)+L-nc}{1-n}\right)-u\left(\frac{L-nc}{1-n}\right)\right]dn+\int_{n_2^*}^{n^{**}}u\left(\frac{R\left(1-L-\frac{nc-L}{r}\right)}{1-n}\right)dn}\times$$
$$\times\left[\int_{n^{**}}^1 u'(c^B(n))\frac{1-r}{n}dn+\frac{n^{**}-\lambda}{\lambda}u'(c)+\right.$$
$$+\sigma_2^*\left[\int_\lambda^{n_2^*}u'(c_L(R,n))\frac{R-1+\frac{n}{\lambda}}{1-n}dn+\int_{n_2^*}^{n^{**}}u'(c_L^D(R,n))\frac{R\left(\frac{n}{\lambda}-1+r\right)}{r(1-n)}dn\right]+$$
$$\left.+(1-\sigma_2^*)\int_\lambda^{n_2^*}u'(c_L(0,n))\frac{\frac{n}{\lambda}-1}{1-n}dn\right]. \tag{96}$$

This expression is positive because $n\geq\lambda$ and $\sigma_2^*\leq 1$. We rearrange (37) and rewrite:

$$\begin{aligned}
&\left[\frac{\partial \sigma_2^*}{\partial L}+\frac{1}{\lambda}\frac{\partial \sigma_2^*}{\partial c}\right]\Delta U(c,L)-\sigma_2^*(1-r)u'(L+r(1-L))=\\
&=\frac{\Delta U(c,L)}{DEN_2}\int_{n^{**}}^1 u'\left(\frac{L+r(1-L)}{n}\right)\frac{1-r}{n}dn-\sigma_2^*(1-r)u'(L+r(1-L))+\cdots=\\
&=\lim_{\iota\to 0}\frac{\Delta U(c,L)}{DEN_2}\left[\int_{n^{**}}^{1-\iota}u'\left(\frac{L+r(1-L)}{n}\right)\frac{1-r}{n}dn+\int_\iota^1 u'\left(\frac{L+r(1-L)}{n}\right)\frac{1-r}{n}dn\right]+\\
&-\sigma_2^*(1-r)u'(L+r(1-L))+\cdots=\\
&=\lim_{\iota\to 0}\frac{\Delta U(c,L)}{DEN_2}\int_{n^{**}}^{1-\iota}u'\left(\frac{L+r(1-L)}{n}\right)\frac{1-r}{n}dn+\\
&+(1-r)u'(L+r(1-L))\left[\frac{\Delta U(c,L)}{DEN_2}-\sigma_2^*\right]+\ldots,
\end{aligned} \tag{97}$$

where the remaining terms are positive, as proved in (96). Hence, (97) is positive if $\Delta U(c,L)-NUM_2\geq 0$. The area inside the dashed line of Figure 10 represents NUM_2, and is clearly smaller than $(1-\lambda)u(c)$.[17] Hence, to prove that $\Delta U(c,L)\geq NUM_2$, it is sufficient to prove that $\Delta U(c,L)\geq(1-\lambda)u(c)$. As $u(c)<\sigma_2^*u(c_L(R))+(1-\sigma_2^*)u(c_L(0))$ by definition of $\underline{\sigma}_2$, a sufficient condition for $\Delta U(c,L)\geq(1-\lambda)u(c)$ is that:

$$\lambda u(c)>u(L+r(1-L)). \tag{98}$$

As $c>L+r(1-L)$, this condition is always satisfied if λ is sufficiently large.

[16]To save on notation, in what follows we do not label the equilibrium values with the superscript BE.
[17]This would hold even if $c_L(0)\geq L+r(1-L)$.

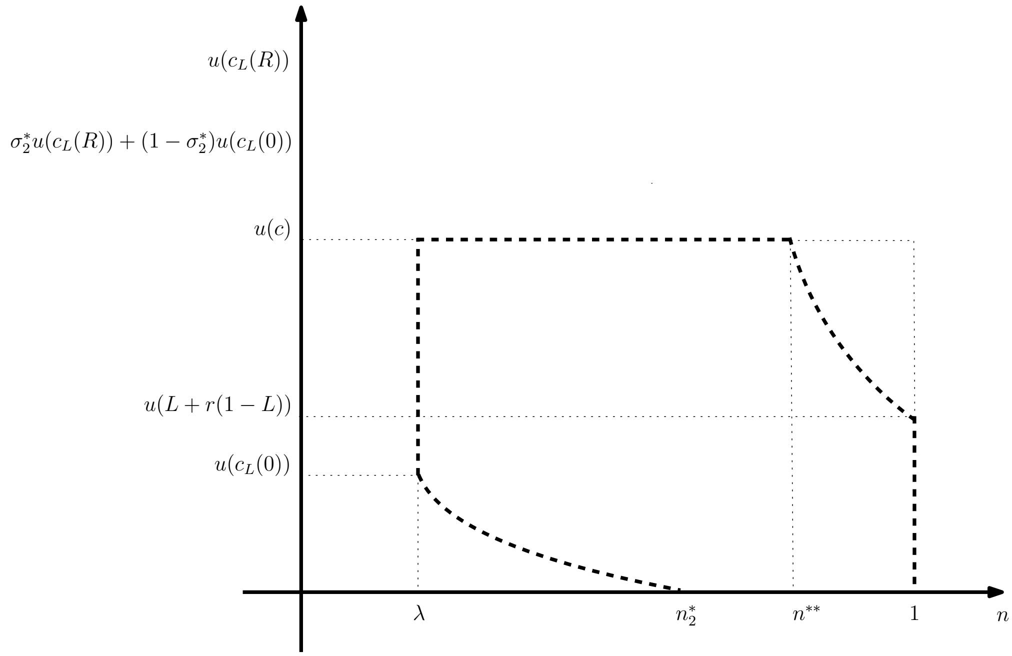

Figure 10: The condition under which $(1-\lambda)u(c) > NUM_2$.

We follow a similar procedure for the pecking order {Liquidation; Liquidity}.

$$\left[\frac{\partial \sigma_1^*}{\partial L} + \frac{1}{\lambda}\frac{\partial \sigma_1^*}{\partial c}\right] = \frac{1}{\int_\lambda^{n_1^*}\left[u\left(\frac{R\left(1-L-\frac{nc}{r}\right)+L}{1-n}\right) - u\left(\frac{L}{1-n}\right)\right]dn} \times$$
$$\times \left[\int_{n^{**}}^{1} u'(c^B(n))\frac{1-r}{n}dn - \int_\lambda^{n_1^*} u'(c_L(0,n))\frac{1}{1-n}dn + \right.$$
$$+ \int_{n_1^*}^{n^{**}} u'(c_L^L(n))\frac{\frac{n}{\lambda}-1+r}{1-n}dn + \left(\frac{n^{**}}{\lambda}-1\right)u'(c) +$$
$$\left. + \sigma_1^* \int_\lambda^{n_1^*}\left[u'(c_L(R,n))\frac{R\left(\frac{n}{r\lambda}+1\right)-1}{1-n} + u'(c_L(0,n))\frac{1}{1-n}\right]dn\right]. \tag{99}$$

This expression is positive because $n \geq \lambda$ and $n^{**} \geq \lambda$. Hence, we rearrange the distorted Euler equation and write:

$$\left[\frac{\partial \sigma_1^*}{\partial L} + \frac{1}{\lambda}\frac{\partial \sigma_1^*}{\partial c}\right]\Delta U(c,L) - \sigma_1^*(1-r)u'(L+r(1-L)) =$$
$$= (1-r)u'(L+r(1-L))\left[\frac{\Delta U(c,L)}{DEN_1} - \sigma_1^*\right] - \int_\lambda^{n_1^*} u'(c_L(0,n))\frac{1}{1-n}dn + \ldots, \tag{100}$$

where the remaining terms are all positive. The previous expression is positive if:

$$\Delta U(c,L) \geq NUM_1 + DEN_1 \int_{\lambda}^{n_1^*} \frac{u'(c_L(0,n))}{u'(L+r(1-L))} \frac{1}{1-r} \frac{1}{1-n} dn. \tag{101}$$

Similarly to the previous case, it can be proved that $NUM_1 < (1-\lambda)u(c)$, and $DEN_1 < (n_1^* - \lambda)$ $u(c_L(R,\lambda))$. Finally, by the coefficient of relative risk aversion being larger than 1 and the definition of $c_L(0,n)$, we can prove that:

$$\int_{\lambda}^{n_1^*} \frac{u'(c_L(0,n))}{u'(L+r(1-L))} \frac{1}{1-r} \frac{1}{1-n} dn < (n_1^* - \lambda) \frac{L+r(1-L)}{L(1-r)}. \tag{102}$$

Hence, a sufficient condition for (101) to hold is:

$$\Delta U(c,L) \geq (1-\lambda)u(c) + (n_1^* - \lambda)^2 u(c_L(R,\lambda)) \frac{L+r(1-L)}{L(1-r)}. \tag{103}$$

By the definition of $\underline{\sigma}_1$ in (20), and the fact that $\underline{\sigma}_1 < \sigma_1^*$, we have:

$$u(c) < \sigma_1^* u\left(\frac{R\left(1-L-\frac{\lambda c}{r}\right)+L}{1-\lambda}\right) + (1-\sigma_1^*)u\left(\frac{L}{1-\lambda}\right) < \sigma_1^* u(c_L(R)) + (1-\sigma_1^*)u\left(\frac{L}{1-\lambda}\right), \tag{104}$$

where the last inequality comes from the definition of $c_L(R)$. Using this expression in the definition of $\Delta U(c,L)$, we can express a sufficient condition for (103) to hold as:

$$\lambda u(c) - u(L+r(1-L)) \geq (n_1^* - \lambda)^2 u(c_L(R,\lambda)) \frac{L+r(1-L)}{L(1-r)} + (1-\sigma_1^*)\left[u\left(\frac{L}{1-\lambda}\right) - u\left(\frac{L-\lambda}{1-\lambda}\right)\right]. \tag{105}$$

The right-hand side of of (105) is positive, and tends to zero as λ tends to 1, as also n_1^* and σ_1^* tend to 1 when λ tends to 1. Thus, as $c > L + r(1-L)$, (105) holds only if λ is sufficiently large.

To sum up, the previous results show under which conditions:

$$\left[\frac{\partial \sigma^{BE}}{\partial L} + \frac{1}{\lambda}\frac{\partial \sigma^{BE}}{\partial c}\right] \Delta U(c,L) - \sigma^{BE}(1-r)u'(L+r(1-L)) > 0. \tag{106}$$

For this to be consistent with (37), it must also be the case that:

$$\int_{\sigma^{BE}}^{1} \left[u'(c) - pRu'(c_L(R))\right] dp > 0, \tag{107}$$

which can be rewritten as:

$$(1-\sigma^{BE})u'(c) - \frac{1-(\sigma^{BE})^2}{2} R u'(c_L(R)) > 0. \tag{108}$$

By the fact that $R(1+\sigma^{BE})/2 > 1$ and the concavity of the utility function, $c < c_L(R)$. Moreover, rearrange the previous expression as:

$$\frac{u'(c)}{u'(c_L(R))} > R\frac{1+\sigma^{BE}}{2} \geq \mathbb{E}[p]R = \frac{u'(c^{PI})}{u'(c_L^{PI}(R))} > 1, \tag{109}$$

where the second inequality holds as $\mathbb{E}[p] = 1/2$ and $\sigma^{BE} \geq 0$, and $\{c^{PI}, c_L^{PI}(R)\}$ is the deposit contract in the equilibrium with perfect information. By the concavity of the utility function, for the ratio $u'(c)/u'(c_L(R))$ to be higher in the banking equilibrium than in the equilibrium with perfect information, it must be the case that $c/c_L(R) < c^{PI}/c_L^{PI}(R)$. To this end, calculate:

$$\frac{\partial}{\partial L}\left[\frac{c}{c_L(R)}\right] = -\frac{(1-\lambda)c}{[R(1-L)+L-\lambda c]^2}(1-R) > 0, \tag{110}$$

$$\frac{\partial}{\partial c}\left[\frac{c}{c_L(R)}\right] = \frac{(1-\lambda)[R(1-L)+L-\lambda c]+\lambda(1-\lambda)c}{[R(1-L)+L-\lambda c]^2} > 0. \tag{111}$$

We take the total differential of the ratio $c/c_L(R)$, evaluated at the equilibrium with perfect information, and look for the condition that makes it negative:

$$\frac{\partial}{\partial L}\left[\frac{c}{c_L(R)}\right] dL + \frac{\partial}{\partial c}\left[\frac{c}{c_L(R)}\right] dc < 0. \tag{112}$$

This implies that:

$$\frac{dL}{dc} < -\frac{\frac{\partial}{\partial c}\left[\frac{c}{c_L(R)}\right]}{\frac{\partial}{\partial L}\left[\frac{c}{c_L(R)}\right]}. \tag{113}$$

As the right-hand side is negative, it must be the case that $dL/dc < 0$. Finally, evaluate the first-order condition with respect to c in (94) at the equilibrium with perfect information. As the term in the integral has to go up when moving from c^{PI} to c^{BE}, then it must be the case that $c^{BE} < c^{PI}$, hence $dc < 0$. This, together with $\frac{dL}{dc} < 0$, implies that $dL > 0$, or $L^{BE} > L^{PI}$. This ends the proof. ■